THE ZERO PARADOX

ZP-A: Lattice Algebra

Version 1.1 | April 2026

Supersedes v1.0 | CC-1 reclassified from Theorem T4

Self-contained within abstract algebra. No topology, probability, or Hilbert space imported. Every claim is provable using only semilattice theory. Cross-framework connections deferred to ZP-E. Version 1.1 change: T4 reclassified as Conditional Claim CC-1. Initialising the sequence at ⊥ is a modelling commitment, not a consequence of A1–A4. Content unchanged; epistemic label corrected.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple:

$$(L, \vee, \bot)$$

where L is a non-empty set (the carrier set of states), $\vee : L \times L \to L$ is a binary operation called join, and $\bot \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom
A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$
A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$
A3 — Idempotency: $x \vee x = x$ for all $x \in L$
A4 — Identity (Additive): $\bot \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes ⊥ the additive identity: the element that contributes nothing to a join and is therefore present in every state as the neutral constituent. This is not a boundary condition or a void — it is the algebraic zero.

II. The Induced Partial Order

2.1 Definition of ≤

Definition D1 — Lattice Order
For x, y ∈ L, define:
$x \leq y :\Longleftrightarrow x \vee y = y$

Theorem T1 — ≤ is a Partial Order
Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓
Antisymmetry: If $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓
Transitivity: If $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 ⊥ is the Least Element

Theorem T2 — ⊥ is a Global Minimum under ≤
For all x ∈ L: $\bot \leq x$
Proof: By A4, $\bot \vee x = x$. By D1, this is the definition of $\bot \leq x$. ✓
T2 is the algebraic statement of the foundational claim: ⊥ is the minimum element that every state sits above. Zero is a constituent of every state is a reading of T2.

III. The Additive Ontology

3.1 No Subtraction Operator

The signature (L, ∨, ⊥) contains no meet operation (∧), no complement, and no subtraction. This is a deliberate structural constraint, not an omission.

Remark R1 — Join-Semilattice vs. Lattice
A full lattice (L, ∨, ∧, ⊥, ⊤) includes a meet operator ∧ and a top element ⊤. The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — removal of informational content. The additive ontology requires that no operation decreases the informational content of a state.

3.2 Join is the Only State Transition

Definition D2 — State Transition
A state transition is any function f: L → L such that $x \leq f(x)$ for all $x \in L$.
Equivalently, $f(x) = x \vee \alpha$ for some $\alpha \in L$.

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence
A state sequence is S: $\mathbb{N} \to L$, written $(S_0, S_1, S_2, \ldots)$, such that:
$S_{n+1} = S_n \vee \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Theorem T3 — State Sequences are Monotone
For any state sequence (S_n) satisfying D3: $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$
Proof: By D3, $S_{n+1} = S_n \vee \alpha_n$. By D1, $S_n \leq S_n \vee \alpha_n$ iff $S_n \vee (S_n \vee \alpha_n) = S_n \vee \alpha_n$. By A1, $(S_n \vee S_n) \vee \alpha_n = S_n \vee \alpha_n$. By A3, $S_n \vee S_n = S_n$. Therefore $S_n \vee \alpha_n = S_{n+1}$. ✓
Monotonicity is a theorem, not a postulate. Derived from A1–A3 via D3.

4.2 The Initial State

Conditional Claim CC-1 — $S_0 = \bot$ [Reclassified from Theorem T4 in v1.0]
Condition: Assume the state sequence is initialised at the minimum of L.
Claim: Under this condition, $S_0 = \bot$, and by T2 and T3: $\bot \leq S_0 \leq S_1 \leq S_2 \leq \ldots$
Corollary: Every state S_n satisfies $\bot \leq S_n$, confirming $\bot$ is a constituent of every state in the sequence.
Reclassification note: In v1.0 this was labelled Theorem T4. The initialisation condition is a modelling commitment, not a consequence of A1–A4. CC-1 makes this explicit. Content unchanged; epistemic status corrected.

V. Open Question — Sufficiency of Monotonicity

Open Question OQ-A1
Is monotonicity (T3) sufficient to characterise all valid state sequences, or are additional axioms required?
OQ-A1a: Is there algebraic reason to restrict α_n beyond membership in L? Should α_n be join-irreducible?
OQ-A1b: Does the open-ended semilattice (without top ⊤) permit unbounded ascending chains? Is that the intended model?
Resolution: OQ-A1 is closed by ZP-E Theorem T5 (Iterative Forcing). The constraint on α_n is existential, not informational: it comes from AX-B1.

Remark R2 — Relationship to ZP-C and ZP-E
OQ-A1 is partially answerable within ZP-A via join-irreducibility. The selection mechanism for α_n is informational/existential and belongs to ZP-C and ZP-E. ZP-E T5 closes OQ-A1 without requiring an algebraic constraint.

VI. Boundary Conditions for ZP-B and ZP-C

Export	Status	Receiving Document
(L, ∨, ⊥) join-semilattice	Derived (A1–A4)	ZP-D: algebraic structure
≤ partial order (D1, T1)	Derived	ZP-D: ordering on states
Monotonicity (T3)	Derived from A1–A3	ZP-D: state layer ordering
⊥ as global minimum (T2, CC-1)	Derived / Conditional	ZP-E: ontological grounding claim
No subtraction / additive ontology (R1)	Structural — signature restriction	ZP-C: no operation reduces content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5	ZP-E owns resolution

VII. Validation Status

Component	Status / Notes
Join-semilattice axioms (A1–A4)	Valid — Axioms; self-contained
Partial order ≤ (D1, T1)	Valid — Derived from A1–A3
⊥ as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1–A3 and D3
CC-1: S_0 = ⊥ (reclassified)	Conditional Claim — modelling commitment; not derived from A1–A4
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5