THE ZERO PARADOX

ZP-B: p-Adic Topology

Version 1.2 | April 2026

Supersedes v1.1 | T0 strengthened with MP-1; C2 forward citation fixed; C3 ordering corrected; T0 Step 6 moved to R2

Self-contained within p-adic analysis and topology. No algebra from ZP-A, no probability, no Hilbert space imported. Version 1.2 changes: (1) MP-1 (Minimality Principle) added as explicit bridge between AX-B1 and the field selection; T0 five-step uniqueness proof now rigorous. (2) C2 derived from T2 only, eliminating forward reference to T5. (3) T5 proven before C3 (formerly T4). (4) Topological irreversibility reclassified from Theorem T4 to Corollary C3. (5) Step 6 of v1.1 T0 moved to motivational Remark R2.

I. The Foundational Distinction

1.1 The Binary Existence Axiom

Axiom AX-B1 — Binary Existence
The foundational distinction of the Zero Paradox framework is binary: a state either exists or it does not. There is no third option at this level.
0 — non-existence (the Null State, corresponding to ⊥ in ZP-A)
1 — existence (the First Atomic State, the minimal non-zero element)
Status: AXIOM. The only non-topological commitment in ZP-B. Precedes p-adic analysis and is the premise from which the field selection is derived.
Scope: AX-B1 asserts the structure of the ontological distinction, not its physical realisation. AX-B1 is invariant across all instantiations.

1.2 The Minimality Principle

The v1.1 derivation showed Q_2 is consistent with AX-B1 because its coefficient set is {0,1}. This is a consistency argument, not a uniqueness argument. A bridge principle is required.

Principle MP-1 — Minimality of Representation

The representational base of the framework must be the minimum base capable of encoding the ontological distinction of AX-B1 without redundancy and without information loss.

Redundancy: A base $p >$ minimum introduces representational states with no ontological counterpart. These carry no meaning and violate parsimony.

Information loss: A base $p <$ minimum cannot distinguish all ontologically distinct states. This violates faithfulness.

Status: PRINCIPLE. MP-1 is a methodological commitment, not a derived result. It is the bridge between the ontological claim of AX-B1 and the mathematical selection of a field.

1.3 Derivation of p = 2

Theorem T0 — p = 2 is the Unique Minimum Sufficient Representational Base

Given: AX-B1 and MP-1. Conclusion: The appropriate p-adic field is Q_2.

Step 1 — Count ontological states. AX-B1 establishes exactly two: non-existence (0) and existence (1).

Step 2 — Minimum sufficient base. A p-adic field Q_p uses coefficients from {0, 1, …, p−1}. Minimum p for exactly two distinct values without redundancy is $p = 2$, coefficient set {0,1}.

Step 3 — Rule out $p = 1$. Base 1 has only one coefficient {0}. Cannot distinguish existence from non-existence. Fails faithfulness (MP-1).

Step 4 — Rule out $p > 2$. For prime $p > 2$, coefficients {2, 3, …, p−1} have no counterpart in AX-B1. Representational states without ontological referents — redundancy (MP-1).

Step 5 — Uniqueness. $p = 2$ is the unique prime satisfying both conditions simultaneously: faithful and non-redundant. Therefore Q_2 is the unique p-adic field consistent with AX-B1 and MP-1. ✓

Status: DERIVED from AX-B1 and MP-1. OQ-B1 closed.

Remark R2 — Binary Decomposition and the Structure of Q_2 (Motivational)

Any composite representational structure built on the binary ontological distinction decomposes, at its foundational level, into binary choices. The hierarchy of Q_2 — with binary branching at every level of the ball structure — reflects this. This is consistent with AX-B1 and reinforces the appropriateness of Q_2, but is not required for the uniqueness proof in T0 Steps 1–5. It is motivational context only.

II. The 2-Adic Field

2.1 The 2-Adic Absolute Value

Fix $p = 2$ (derived in T0). Every non-zero rational $q \in \mathbb{Q}$ can be written uniquely as $q = 2^v \cdot (a/b)$ where $v \in \mathbb{Z}$ and a, b are not divisible by 2. The integer v is the 2-adic valuation $v_2(q)$. By convention, $v_2(0) = +\infty$.

Definition D1 — 2-Adic Absolute Value
For $q \in \mathbb{Q}$: $\lvert q\rvert_2 = 2^{-v_2(q)}$ for $q \neq 0$; $\lvert 0\rvert_2 = 0$
Elements with high powers of 2 are considered small under $\lvert\cdot\rvert_2$. Elements with no factor of 2 have $\lvert\cdot\rvert_2 = 1$.

2.2 The 2-Adic Field Q_2

Q_2 is the completion of $\mathbb{Q}$ under the metric induced by $\lvert\cdot\rvert_2$. Elements of Q_2 are formal power series in 2:

$$x = \Sigma_{n=v}^{\infty} a_n \cdot 2^n \text{ where } a_n \in \{0,1\} \text{ and } v = v_2(x) \in \mathbb{Z}$$

The coefficients $a_n \in \{0,1\}$ are precisely the binary values of AX-B1. The element $0 \in Q_2$ has $v_2(0) = +\infty$ and $\lvert 0\rvert_2 = 0$.

2.3 The Minimum Viable Deviation — ε_0

Definition D5 — Minimum Viable Deviation ε_0
$\varepsilon_0 = 2^k$ for some integer k, where k is the maximum valuation accessible in the instantiation.
Status: DEFINED — universe-contingent parameter.
Structural role (universal): ε_0 is always the first element crossed by the Snap. This role is fixed by the structure of Q_2 and AX-B1.
Numerical value (contingent): The specific value of k is determined by the physical constants of the instantiation. Planck-scale quantities are candidates in our universe.

III. The Ultrametric

Definition D2 — 2-Adic Metric
For $x, y \in Q_2$: $d(x, y) = \lvert x - y\rvert_2$

Theorem T1 — Strong Triangle Inequality (Ultrametric)

For all x, y, z ∈ Q_2: d(x, z) ≤ max(d(x, y), d(y, z))

Proof: Write x − z = (x − y) + (y − z). The ultrametric property of v_2 gives $v_2(a+b) \geq \min(v_2(a), v_2(b))$ for all a, b, from which $|a+b|_2 \leq \max(|a|_2, |b|_2)$. Apply with a = x−y and b = y−z. ✓

This is strictly stronger than the ordinary triangle inequality. The ultrametric forces a non-Archimedean geometry in which any point inside a ball can serve as its centre.

Corollary C1 — All Triangles are Isosceles

If d(x,y) ≠ d(y,z), then d(x,z) = max(d(x,y), d(y,z)).

Proof: Suppose d(x,y) < d(y,z). By T1, d(x,z) ≤ d(y,z). Also d(y,z) ≤ max(d(y,x), d(x,z)) = d(x,z) since d(x,y) < d(y,z). Therefore d(x,z) = d(y,z). ✓

Definition D3 — Ball in Q_2

B(a, r) = { x ∈ Q_2 : d(x, a) ≤ r } (closed ball)

B°(a, r) = { x ∈ Q_2 : d(x, a) < r } (open ball)

Theorem T2 — Every Ball is Clopen

Proof (closed ball is open): Let y ∈ B(a, r). For any z ∈ B(y, r), T1 gives d(z, a) ≤ max(d(z,y), d(y,a)) ≤ r. So B(y,r) ⊆ B(a,r). Every point is an interior point. ✓

Proof (open ball is closed): Let $(x_n) \to x$ with all x_n ∈ B°(a,r). Ball radii in Q_2 are discrete (powers of 2), so d(x,a) < r holds in the limit. ✓

Corollary C2 — Disjoint Balls Do Not Communicate [v1.2: derived from T2 only]
If B(a, r) and B(b, r) are disjoint, no continuous path exists from any point in B(a, r) to any point in B(b, r).
Proof: By T2, B(a, r) is clopen. Any continuous f: [0,1] → Q_2 with f(0) ∈ B(a,r) and f(1) ∈ B(b,r) would require f to map the connected set [0,1] onto a subset intersecting both B(a,r) and its clopen complement. The preimage of a clopen set under a continuous function is clopen in [0,1]. Since [0,1] is connected, this preimage is either empty or all of [0,1] — neither is possible. Contradiction. ✓
Note v1.2: The v1.1 proof cited T5 before T5 was proven. This proof uses only T2 and the connectedness of [0,1]. No forward reference.

IV. Topological Isolation of Zero

Theorem T3 — Topological Isolation of 0
For $r = 2^{-k}$, the ball B(0, r) = { x ∈ Q_2 : $v_2(x) \geq k$ }. Any x outside this ball has $d(0,x) \geq 2^{-k+1}$ > r. B(0,r) and its complement are separated by a gap of at least 2^{-k}.
The transition from 0 to any non-zero element is a discrete jump across a clopen boundary — the topological identity of the Snap.
Relationship to ε_0: $\varepsilon_0 = 2^k$ is the smallest non-zero element outside the tightest ball around 0. The Snap crosses exactly this gap.

V. Topological Structure of Q_2

5.1 Total Disconnectedness — proven before C3

Theorem T5 — Q_2 is Totally Disconnected
The only connected subsets of Q_2 are singletons.
Proof: Let S ⊆ Q_2 contain two distinct points a, b with d(a,b) = r > 0. Choose s with 0 < s < r. By T2, B(a,s) is clopen. Then S = [S ∩ B(a,s)] ∪ [S \ B(a,s)] is a separation of S into two disjoint non-empty clopen sets. S is not connected. Since S was arbitrary, only singletons are connected. ✓

Definition D4 — Topological Irreversibility
A transition from a to b in a topological space X is topologically irreversible if there exists no continuous path $\gamma: [0,1] \to X$ with $\gamma(0) = b$ and $\gamma(1) = a$.

Corollary C3 — The Snap is Topologically Irreversible [reclassified from Theorem T4 in v1.1]
Let $x \in Q_2$ with $x \neq 0$. There exists no continuous path $\gamma: [0,1] \to Q_2$ with $\gamma(0) = x$ and $\gamma(1) = 0$.
Proof: By T5, Q_2 is totally disconnected. A path with endpoints $x \neq 0$ and 0 would require a connected subset containing two distinct points. By T5, none exists. ✓
Reclassification note: This is a direct corollary of T5. Derivation chain: T1 → T2 → T5 → C3. Result unchanged; classification corrected.
Scope: Statement about continuity in the 2-adic topology only. Thermodynamic irreversibility is a separate claim not developed in this document.

VI. Universal Structure vs. Contingent Parameters

Remark R1 — Universal Structure vs. Universe-Contingent Parameters
Universal (invariant across all instantiations): AX-B1 (binary existence, a logical fact); MP-1 (minimality, a methodological commitment); T0 (p=2 from AX-B1+MP-1); T1, T2, T3, T5, C1, C2, C3 (all topological results); structural role of ε_0.
Universe-contingent: Numerical value of ε_0 (determined by physical constants; Planck-scale quantities in our universe). Any physical prediction invoking ε_0 numerically inherits this contingency.
Consequence: The Zero Paradox is a universal ontology of state emergence, not a physical theory of our universe specifically.

VII. Validation Status

Component	Status / Notes
AX-B1: Binary Existence Axiom	Axiom — explicit; load-bearing premise
MP-1: Minimality Principle	Principle — explicit bridge; resolves v1.1 gap in T0
T0: p = 2 from AX-B1 + MP-1	Valid — Derived; five-step uniqueness proof; Step 6 moved to R2
T1: Strong triangle inequality	Valid — Derived
C1: All triangles isosceles	Valid — Corollary of T1
T2: Every ball is clopen	Valid — Derived from T1
C2: Disjoint balls do not communicate	Valid — Derived from T2 only; forward citation removed
T3: Topological isolation of 0	Valid — Derived from D1 and D2
T5: Q_2 totally disconnected	Valid — Derived; proven before C3
C3: Snap topologically irreversible	Valid — Corollary of T5; reclassified from Theorem T4
D5: ε_0 minimum viable deviation	Valid — Defined; structural role universal; value contingent
R2: Binary decomposition	Valid — Remark; motivational context; not part of T0 proof