THE ZERO PARADOX

ZP-H Addendum

The Snap Floor in Native Categories

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ZP-H derives the Binary Snap across four mathematical frameworks and locates the snap floor ⊥ as the categorical bottom in each. In the main document that location is carried by three depth-index proxy categories — copies of ℕ with the order ≤ (Q_2BallDepth, InfoDepth, HilbDimDepth), engineered so that depth 0 is the initial object. This addendum does the stronger thing: it drops the snap floor into each framework's own category, taken directly from Mathlib, where the floor appears in that field's native syntax. A topologist, a functional analyst, and a probabilist each meet the same bottom, each in their own category and their own language.

Three genuine Lean functors realize the snap chain inside the standard categories: F_B into TopCat (topological spaces), F_D into ModuleCat ℂ (ℂ-modules), and F_C into the Kleisli category of the probability monad PMF (stochastic maps). Each is sorry-free, and each carries the snap floor to its category's categorical bottom — an inverse limit in TopCat, an initial object in ModuleCat ℂ and in the Kleisli category. The bundled witness is mc1_correspondence (ZPH_MC1.lean).

Two honesty boundaries hold throughout. The realized claim is correspondence, not identity. What is proved is that each bottom is its own category's categorical bottom, agreeing on the snap; that the four bottoms are numerically one object remains a modeling commitment, not a theorem. The native categories are the honest target. Mathlib has no bespoke category of p-adic spaces, Hilbert spaces, or information spaces, so the general-purpose categories — topological spaces, ℂ-modules, stochastic maps — are where the floor genuinely lives.

Section I: From Depth Proxies to Native Categories

The proxy categories of the main document are correct but indirect. Q_2BallDepth, InfoDepth, and HilbDimDepth are each a copy of ℕ with the order ≤, wrapped so that the bottom 0 satisfies the universal property of an initial object and the source-asymmetry condition AX-G2 (no morphism returns to 0). They witness the snap floor's categorical role, but inside a structure built for the purpose rather than inside the domain itself.

The native categories are not built for the purpose, and that is the point — they also do not satisfy the ZP categorical axioms. A ZPCategory (ZP-G) requires that no terminal object exist (AX-G1): the snap has nowhere higher to go. TopCat and ModuleCat ℂ both have a terminal object (the one-point space; the zero module). So the snap floor cannot be "the initial object of a ZPCategory" here. The honest statement is the one each category supports natively: in TopCat the floor is the limit of the shrinking system, and in ModuleCat ℂ and the Kleisli category it is the genuine initial object. Each is stated below.

Section II: F_B — The Snap Floor in TopCat

The 2-adic snap floor is $0 \in Q_2$. Around it sits the descending tower of clopen balls $B(0, 2^{-n})$ — the unit ball, then radius 1/2, then 1/4, and so on — each a genuine topological subspace of Q_2. The inclusions $B(0, 2^{-m}) \subseteq B(0, 2^{-n})$ for $n \le m$ are continuous, so the tower is a functor out of $\mathbb{N}^{op}$ (the balls shrink as the index grows): a real diagram in TopCat.

Functor: fB_functor : ℕ^op → TopCat (ZPH_TopFunctor.lean)
Object n ↦ the clopen ball $B(0, 2^{-n}) \subseteq Q_2$, as a TopCat object.
Morphism ($n \le m$): the continuous inclusion of the smaller ball into the larger.
fB_bottom_is_limit: $\cap_n B(0, 2^{-n}) = \{0\}$
The snap floor is the inverse limit of the tower — the one point common to every ball.
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

TopCat has both a terminal object (the one-point space) and an initial object (the empty space); the snap floor is neither. It is the limit of the ball tower — the categorical expression a topologist would expect for "the point everything converges to." This is the honest realization in a category that is not a ZPCategory: not an initial object, but a limit.

Section III: F_D — The Snap Floor in ModuleCat ℂ

The state layer represents the snap in a complex Hilbert space: StateSpace n is the n-dimensional space $\mathbb{C}^n$, with the snap an orthogonal shift between basis states (ZP-D). As ℂ-modules, these spaces and the linear maps between them form a diagram in ModuleCat ℂ.

Functor: fD_functor : ℕ → ModuleCat ℂ (ZPH_HilbFunctor.lean)
Object n ↦ StateSpace n = the ℂ-module $\mathbb{C}^n$.
Morphism ($n \le m$): the isometric ℂ-linear embedding that pads the extra coordinates with zero.
fD_zero_isInitial: StateSpace 0 is the initial object of ModuleCat ℂ.
The snap floor is the zero module — the one-element ℂ-module {0}, which is the categorical zero object (both initial and terminal).
fD_embed_inner: the embeddings preserve the inner product (genuine isometries).
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

Here, unlike TopCat, the floor is the initial object outright: the zero object of ModuleCat $\mathbb{C}$ is StateSpace 0. The isometry lemma records what ModuleCat $\mathbb{C}$ forgets — that the embeddings are not merely linear but inner-product preserving — so the Hilbert-space structure travels alongside the functor rather than being discarded by the choice of category.

Section IV: F_C — The Snap Floor in KleisliCat PMF

The information layer measures the snap as one bit of divergence between distributions (ZP-C, JSD = log 2). Its native category is the Kleisli category of the finite-distribution monad PMF: objects are finite types, and a morphism A → B is a stochastic map (a Markov kernel), assigning to each a ∈ A a probability distribution on B. Mathlib supplies this category for free, since PMF is a lawful monad.

Functor: fC_functor : $\mathbb{N}$ → KleisliCat PMF (ZPH_InfoFunctor.lean)
Object n ↦ Fin n, the type of n distinguishable outcomes.
Morphism (n ≤ m): the deterministic embedding (each outcome mapped to its image with certainty).
fC_zero_isInitial: Fin 0 (the empty type) is the initial object.
fC_no_return: for n > 0 there is NO stochastic map Fin n → Fin 0.
AX-G2 as a theorem: no probability distribution lives on the empty type, so nothing returns to the floor.
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

The empty type is the snap floor here, and it carries the snap's irreversibility for free. A stochastic map into Fin 0 would be a probability distribution on the empty type, and there is none. So while every object has a unique map out of Fin 0 (it is initial), no nonempty object has any map back — AX-G2 (source asymmetry) is not assumed but proved, as fC_no_return. Of the three native categories, this is the one where the framework's irreversibility axiom becomes a theorem of the ambient mathematics.

Section V: The MC-1 Correspondence, Scope, and Purity

The three functors, together with F_A (the join-semilattice $\mathbb{N}$ of the main document, where 0 is already the initial object), are bundled as a single witness.

Definition: mc1_correspondence : MC1Correspondence (ZPH_MC1.lean)
A record collecting the native-category realizations:
• F_D: StateSpace 0 is initial in ModuleCat $\mathbb{C}$.
• F_C: Fin 0 is initial in KleisliCat PMF, with no return morphism.
• F_B: the clopen-ball tower has inverse limit {0} in TopCat.

Definition: mc1_correspondence : MC1Correspondence (ZPH_MC1.lean)

Sorry-free. Purity [propext, Classical.choice, Quot.sound].

This settles the correspondence half of MC-1: across the native categories, the snap floor is each category's own categorical bottom, and the four agree on the snap. It does not settle the identity half. That the algebraic ⊥, the 2-adic 0, the zero module, and the empty type are numerically one object is a modeling commitment — a choice to read four categorical bottoms as a single thing — not a theorem proved here. The same fence stands as for the diagonal fixed point: the framework reads these as faces of one object; it proves each the categorical bottom of its own category, and leaves their literal identity as an interpretive commitment.

Scope and Purity

This is a formalisation milestone, not new category theory. TopCat, ModuleCat ℂ, and the Kleisli category of PMF are standard Mathlib categories with standard universal properties. The contribution is realizing the ZP snap floor inside each as a genuine functor, in place of the ℕ-indexed depth proxies of the main document.

Purity: [propext, Classical.choice, Quot.sound]. Unlike the choice-free ZP-J results, these functors inherit Classical.choice from Mathlib's topology, inner-product, and PMF libraries. The dependency is a library inheritance, not a new commitment of the construction; whether it is structurally forced is the open question tracked under Classical.choice necessity (ZP-L / ZP-M).

Lean Source Files

ZPH_TopFunctor.lean — fB_functor, fB_bottom_is_limit (F_B into TopCat).

ZPH_HilbFunctor.lean — fD_functor, fD_zero_isInitial, fD_embed_inner (F_D into ModuleCat ℂ).

ZPH_InfoFunctor.lean — fC_functor, fC_zero_isInitial, fC_no_return (F_C into KleisliCat PMF).

ZPH_MC1.lean — mc1_correspondence (the bundled witness).

All in ZeroParadox/ in the public repository.

Endnote: This document is an addendum to ZP-H Categorical Bridge and reads after it. ZP-H locates the snap floor as the categorical bottom across four frameworks using depth-index proxy categories; this addendum realizes the same floor inside each framework's native Mathlib category — TopCat, ModuleCat ℂ, and the Kleisli category of distributions. All results sorry-free in Lean 4 as of June 2026, footprint [propext, Classical.choice, Quot.sound].