THE ZERO PARADOX

ZP-J Wheel Addendum

The Wheel of Fractions is a Wheel

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A wheel (Carlström 2001) is an algebraic structure that extends a commutative ring by making division a total operation: every element, including 0, has a reciprocal /x, so /0 becomes a defined first-class element rather than an error. The two elements this produces — $\infty = /0$ (the reciprocal of zero) and $\bot = 0 \cdot /0$ (an absorbing "undefined" element) — are what distinguish a wheel from a field. ZP-J Self-Reference left open which structure the Zero Paradox porthole gives rise to: a wheel, in which ∞ and $\bot$ are distinct, or a meadow, in which they collapse. This addendum settles that question.

The main result is WheelFrac.instWheel (ZPJ_WheelFrac.lean): for any commutative ring A and any multiplicative submonoid S, the wheel of fractions $\odot_S A = (A \times A)/\equiv_S$ satisfies every axiom of Carlström's Definition 1.1. The companion result WheelFrac.inf_ne_bot shows that, whenever $0 \notin S$, the two special elements stay distinct ($\infty \neq \bot$) — so the construction is a wheel, not a meadow. The construction is Carlström's; the contribution here is a faithful, machine-verified encoding of it that is also free of the axiom of choice (footprint [propext, Quot.sound]), situated as the algebraic form of the ZP porthole.

Section I: The Wheel Axioms (Carlström Definition 1.1)

A wheel is a set W with two binary operations + and ·, a unary involution / (reciprocal), and two constants 0 and 1, subject to eight axioms. Carlström's Definition 1.1 packages the additive and multiplicative parts as commutative monoids; the ZP Wheel typeclass unbundles those two monoid axioms into their separate equational laws, giving 14 fields that are equivalent to Carlström's eight. The unbundling is bookkeeping only — no axiom is added, removed, or weakened.

Typeclass: Wheel (ZPJ_Wheel.lean)
class Wheel (W : Type*) where
wadd, wmul : W → W → W -- + and ·
winv : W → W -- the involution /
wzero, wone : W -- 0 and 1
-- W1–W3: (W, +, 0) commutative monoid [Carlström (1)]
-- W4–W6: (W, ·, 1) commutative monoid [Carlström (2), monoid part]

Typeclass: Wheel (ZPJ_Wheel.lean)
-- W7: $/(/x) = x$ [Carlström (2), involution]
-- W8: $/(x·y) = /x · /y$ [Carlström (2), involution]
-- W9: weakened distributivity [Carlström (3)]
-- W10: $(x·/y + z) + 0·y = (x + y·z)·/y$ [Carlström (4)]
-- W11: $0·0 = 0$ [Carlström (5)]
-- W12: $(x + 0·y)·z = x·z + 0·y$ [Carlström (6)]
-- W13: $/(x + 0·y) = /x + 0·y$ [Carlström (7)]
-- W14: $x + 0·/0 = 0·/0$ [Carlström (8)]

The last group of axioms is what makes division total. W7 and W8 make / an involution that distributes over multiplication; W9–W14 govern how the two derived elements $\infty = /0$ and $\bot = 0·/0$ interact with + and ·. A wheel in which $\infty = \bot$ is exactly a meadow: the distinction between the two is the whole content of "wheel, not meadow."

Section II: The Wheel of Fractions ⊙_S A

The wheel of fractions generalises the field-of-fractions construction so that it survives division by zero. Start from a commutative ring A and a multiplicative submonoid $S \subseteq A$ (a set containing 1 and closed under multiplication). Form pairs $(x, y) \in A \times A$ — read as the formal fraction x/y — and quotient by the relation $\equiv_S$ below.

Construction (Carlström 2001, pp. 4–5)
$\odot_S A = (A \times A) / \equiv_S$
$(x, y) \equiv_S (x', y') \Leftrightarrow \exists\, s, s' \in S,\ s·x = s'·x' \wedge s·y = s'·y'$
$0 = [0, 1]$ $1 = [1, 1]$
$[x,y] + [x',y'] = [x·y' + x'·y,\ y·y']$
$[x,y] · [x',y'] = [x·x',\ y·y']$
$/[x,y] = [y, x]$ (the involution is pair-swap)
Then $/0 = [1,0] = \infty$ and $0·/0 = [0,0] = \bot$.

The choice of relation matters. The naive cross-multiplication $x·y' = x'·y$ — the one that defines equality of ordinary fractions — is not an equivalence relation on a general commutative ring: transitivity fails without a cancellation law. The submonoid-quotient relation $\equiv_S$ repairs this by witnessing each identification with elements of S, and it is provably reflexive, symmetric, and transitive (WheelFrac.srel). Each of the five operations is then well-defined on the quotient — the proofs that they respect $\equiv_S$ are the bulk of the formalisation.

Section III: ⊙_S A is a Wheel

With the operations shown well-defined, every one of the 14 typeclass fields is discharged. The additive and multiplicative monoid laws and the distributivity and porthole axioms all reduce, after lifting representatives, to ring identities in A closed by ring normalisation; the two involution laws (W7, W8) hold definitionally because pair-swap is its own inverse and commutes with the componentwise product.

Theorem: WheelFrac.instWheel (ZPJ_WheelFrac.lean)
∀ {A : Type*} [CommRing A] (S : Submonoid A),
Wheel ($\odot_S$ A)
For every commutative ring A and multiplicative submonoid S, the wheel of fractions $\odot_S$ A satisfies all 14 fields of the ZP Wheel typeclass — equivalently, all eight axioms of Carlström's Definition 1.1.
Sorry-free. Lean purity: [propext, Quot.sound] — Classical.choice-free.

Section IV: Wheel, Not Meadow

A wheel collapses to a meadow precisely when its two special elements coincide. For the wheel of fractions this collapse happens exactly when $0 \in S$: if S contains a zero divisor witness for 0, the fraction [1,0] and the fraction [0,0] become identified. The natural hypothesis $0 \notin S$ (which holds whenever S is the complement of a prime ideal, the usual case) keeps them apart.

Theorem: WheelFrac.inf_ne_bot (ZPJ_WheelFrac.lean)
∀ {A : Type*} [CommRing A] (S : Submonoid A),
$(0 : A) \notin S \to \infty \neq \bot$ in $\odot_S$ A
When $0 \notin S$, the reciprocal of zero ($\infty = /0$) and the absorbing element ($\bot = 0 \cdot /0$) are distinct. The construction is a genuine wheel, not a meadow.
Proof: $\infty = \bot$ would yield witnesses $s, s' \in S$ with $s \cdot 1 = s' \cdot 0$, forcing $s = 0$ and hence $0 \in S$ — contradicting the hypothesis.
Sorry-free. Lean purity: [propext, Quot.sound] — Classical.choice-free.

Remark — The Porthole Connection

In ZP-J, the porthole is the point where val(⊥) = ∞ and ⊥ = {⊥} coincide — the same structural fact written in the 2-adic valuation ($v_2(0) = \infty$) and in ZF+AFA (the Quine atom). The wheel of fractions is the algebraic face of that point: /0 is defined and distinct from the absorbing ⊥, exactly the behaviour the porthole predicts. The concrete carrier ZPWheelElem (ZPJ_Wheel.lean §III–VI) makes this explicit on the rationals extended with ∞ and ⊥, where val(x) = ∞ ⇔ /x = ∞ is proved directly (zpw_top_val_iff_inv_is_inf). This addendum's headline result is the general construction over an arbitrary commutative ring; the concrete carrier is the illustrative special case.

Section V: Scope, Purity, and Relationship to Carlström

Two scope boundaries are worth stating plainly. This is a formalisation, not new mathematics. The wheel of fractions and the theorem that it is a wheel are Carlström's. What is contributed here is a machine-checked encoding faithful to Definition 1.1, with a verified axiom footprint, placed in the ZP porthole context. Ring structure is an input, not a conclusion. The construction starts from a commutative ring and a submonoid; it does not derive wheel structure from the ZP lattice axioms alone. The bridge typeclass that would state such a derivation, WheelValuationStructure (ZPJ_Wheel.lean §VII), is defined but its porthole condition val(0) = ⊤ is an assumed axiom, motivated by the ZP argument rather than type-checked as necessary.

Lean Source Files
ZPJ_WheelFrac.lean — rel, srel, the five quotient operations (waddF, wmulF, winvF), instWheel, inf_ne_bot. The headline results of this addendum.
ZPJ_Wheel.lean — the Wheel typeclass (Carlström Def 1.1, 14 fields), the derived elements wheelInf / wheelBot, the concrete carrier ZPWheelElem, and the porthole correspondence zpw_top_val_iff_inv_is_inf.
Both files in ZeroParadox/ in the public repository.

Axiom Footprint (headline results: instWheel, inf_ne_bot)
[propext, Quot.sound] — Classical.choice-free.
propext — propositional extensionality (standard in Lean 4)
Quot.sound — quotient soundness (standard in Lean 4; the construction is a quotient, so this is expected and unavoidable)
No Classical.choice. No Dependent Choice. No set-theoretic assumptions.

Remark R-J.W — Relationship to Carlström's Theorem

Carlström introduced wheels in Wheels — On Division by Zero (Research Reports in Mathematics No. 11, Department of Mathematics, Stockholm University, 2001; a Licentiate thesis), where Definition 1.1 (p. 5) gives the eight wheel axioms and the wheel of fractions is constructed (§1.2, §4.2). The work was later published as Wheels — on division by zero, Mathematical Structures in Computer Science 14(1):143–184, 2004. Carlström proved there that the wheel of fractions of a commutative ring is a wheel. The result here is that theorem, encoded in Lean 4 against a typeclass that reproduces his Definition 1.1 field for field, and discharged without the axiom of choice. The encoding is what is new: a third party can read the 14 fields against Carlström's eight axioms and confirm the correspondence, and can read the axiom footprint to confirm the choice-free claim. Whether the porthole condition $\mathrm{val}(0) = \top$ can be derived — rather than assumed — from upstream ZP structure remains the open question flagged in ZPJ_Wheel.lean §VII–VIII.

Endnote: This document is an addendum to ZP-J Self-Reference and reads after it. ZP-J established the porthole ($\mathrm{val}(\bot) = \infty$, $\bot = \{\bot\}$); this document gives its algebraic form, the wheel of fractions, and the machine-verified proof that the construction is a wheel rather than a meadow. All results sorry-free in Lean 4 as of June 2026, footprint [propext, Quot.sound].