ZP-L Illustrated Companion

The Ordinal Summit

Where the Tower Snaps to c_1

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This companion explains the ideas in plain language with diagrams and examples. It is not the formal document — every claim here restates a result already proved in ZP-L Incomputability Convergence. Consult that document for the authoritative mathematics.

What Is ZP-L Doing?

ZP-K proved that the initial machine state c_0 is a Kleene fixed point: a program whose behavior is determined entirely by its own index, guaranteed to exist by the second recursion theorem. ZP-L picks up the thread from a different direction: the ordinal direction. If a map ϕ assigns machine phases to ordinals — sending c_0 to everything below a certain point and c_1 above it — what is that threshold ordinal? ZP-L establishes the answer: ε_0, the first fixed point of the map $\alpha \mapsto \omega^\alpha$.

This is not an arbitrary choice. ZP-L proves that ε_0 is the minimal threshold — no monotone, tower-aligned map can snap before ε_0, and the canonical map snaps exactly there. ZP-L also establishes the connection to ZP-B: as the tower stages approach ε_0 in ordinal order, their 2-adic encodings converge to $0 = \bot$ in $\mathbb{Z}_2$. The same sequence witnesses both convergences simultaneously.

Real-world analogy — the unreachable horizon

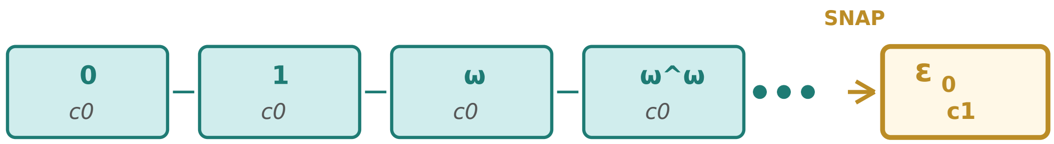
Imagine a path where each step is longer than the last: you walk 1 mile, then 2, then 4, then 8, ... You are always getting further from the start, but you are also always somewhere on the path — never at the horizon. The horizon (ε_0) is the limit of all those steps combined. It exists as a mathematical object, but no finite number of steps reaches it.

The ordinal tower works the same way. The snap that ZP-E proved occurs at ε_0 is not a step you can take from inside the tower. It is the point where the tower stops and something new begins.

What Is ε_0?

ε_0 (epsilon-zero) is the smallest ordinal α satisfying the equation $\omega^\alpha = \alpha$. In other words, it is the first fixed point of the map that sends any ordinal α to ω^α (omega raised to the power α). This is not a coincidence or a definition by fiat — it is a property ε_0 satisfies and no smaller ordinal does.

The tower $0, 1, \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ climbs through the ordinals, each stage being ω raised to the previous stage. Every stage is strictly below ε_0, and ε_0 is their limit — the supremum of the whole sequence. It is never reached from below; it is the boundary above all stages.



Each tower stage is strictly below ε-zero and maps to c0. The snap to c1 occurs exactly at ε-zero.

The tower stages 0, 1, ω, ω^ω, ... all lie strictly below ε_0 and map to c_0. The snap transition to c_1 occurs at the threshold ε_0 itself. The arrow marks the snap. ZP-L §VII: epsilon_zero_snap_canonical.

> Remember: ε_0 is not "infinity." It is a specific countable ordinal — the first one satisfying $\omega^\wedge\alpha = \alpha$. Ordinals larger than ε_0 exist ($\varepsilon_0 + 1$, ε_1, and many more). ZP-L is not claiming that ε_0 is the largest ordinal or that the framework breaks down above it. It is the minimal snap threshold for monotone, tower-aligned maps — and that minimality is exactly what is proved.

Why ε_0 Is the Exact Threshold

Two facts together pin ε_0 as the minimal snap threshold. They work like a lower bound and an upper bound that meet at the same point.

Lower bound (snap cannot happen before ε_0): The fundamental sequence is cofinal in ε_0. This means: for any ordinal α below ε_0, some tower stage eventually exceeds α. A monotone map that assigns c_0 to every tower stage must therefore assign c_0 to α too (it cannot increase between a stage above α and α itself). So no ordinal strictly below ε_0 can be a snap point for such a map.

Upper bound (snap does happen at ε_0): ε_0 is itself a fixed point of $\omega^\wedge\cdot$ ($\omega^\wedge\varepsilon_0 = \varepsilon_0$). A map that assigns c_1 to all fixed points of $\omega^\wedge\cdot$ must assign c_1 to ε_0. Together with the lower bound, ε_0 is the first place where c_1 can and must appear.

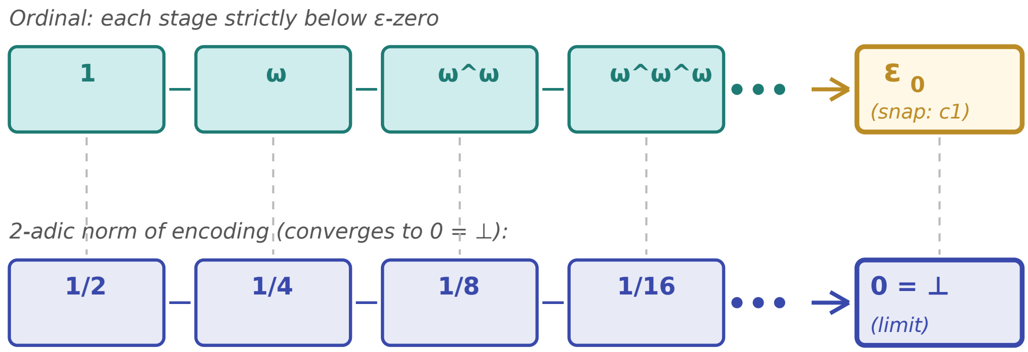
The canonical witness map $\phi(\alpha) = c_0$ if $\alpha < \varepsilon_0$, else c_1 satisfies all five conditions at once — monotonicity, tower-alignment, fixed-point-respecting, snapping at ε_0, and minimality — with no free hypotheses. This is the central result of ZP-L.

> "Minimal" does not mean "unique." ZP-L proves that no snap can occur strictly before ε_0 (under the stated conditions), not that ε_0 is the only possible snap point. Maps that also assign c_1 to ordinals above ε_0 are not ruled out. The framework fixes ε_0 as the threshold by the canonical witness, not by uniqueness of all satisfying maps.

The Two-Adic Connection

ZP-B established that $0 = \bot$ in the 2-adic integers $\mathbb{Z}_2$ is the valuative sink: the 2-adic valuation of 0 is infinite, which means 0 sits at the bottom of the 2-adic metric. ZP-L connects this to the ordinal tower via an encoding of ordinals below ε_0 into $\mathbb{Z}_2$ using their Cantor normal forms.

The encoding (cnfToZp2 in ZPL.lean) maps the n-th tower stage to 2^n in $\mathbb{Z}_2$. The 2-adic norm of 2^n is $(1/2)^n$, which goes to 0 as n increases. So as the ordinal tower climbs toward ε_0 from below, its 2-adic image converges to $0 = \bot$ in the 2-adic metric (the norms $(1/2)^n$ decrease monotonically and converge to 0). Two convergences, one tower.



Same tower, two perspectives. Teal (top): ordinal stages approach ε-zero. Indigo (bottom): their 2-adic norms approach 0.

Same tower stages, two perspectives. Teal (top row): ordinal stages 1, ω, ω^ω, ... approach ε_0 from below. Indigo (bottom row): the 2-adic norms of their encodings 1/2, 1/4, 1/8, ... converge to $0 = \bot$. ZP-L §VII: snap_zp2_correspondence.

This dual convergence is what snap_zp2_correspondence formalizes in ZP-L §VII: four jointly established facts about the same tower sequence — all stages below ε_0 in ordinal order, all assigned c_0 by the canonical map, all encoding to 2^n in $\mathbb{Z}_2$ with 2-adic norm $(1/2)^n \to 0$, and the canonical map snapping to c_1 at ε_0.

The Kleene Connection

ZP-K identified ⊥ as a Kleene fixed point: a program that is its own output, guaranteed to exist by Kleene's second recursion theorem. ZP-L identifies ε_0 as an ordinal fixed point: the first ordinal satisfying ω^$\alpha = \alpha$, guaranteed to exist by the fixed-point theorem for normal functions. Both fixed-point proofs require Classical.choice in their Lean formalizations. In each case, the choice axiom enters at a non-constructive selection step: the ordinal proof uses it for the supremum that defines ε_0; the Kleene proof uses it for the fixed-point selection. Whether these reflect the same underlying mathematical phenomenon — or are coincidentally parallel — is the open question ZPL.lean §I poses.

ZP-L §VI (kleene_ordinal_snap_bridge) names this structural parallel explicitly. The theorem itself is purely ordinal — no Code or eval object appears in it. What the section establishes is that the hypothesis structure of the two proofs is the same: diagonalization, non-constructive selection, fixed-point existence.

Two rooms, same pattern

Imagine two mathematicians working in separate rooms. One (ZP-K) is computing with programs and asking: what program runs itself? The other (ZP-L) is computing with ordinals and asking: what ordinal satisfies ω^α = α? Neither knows what the other is doing. When they compare notes, they find the same proof structure: the same diagonal argument, the same non-constructive selection step, the same kind of fixed-point existence result.

ZPM formalizes this parallel with a type bridge between the two settings. The two rooms share the same floor plan.

Convergence with Proof Theory

ZP-L derives ε_0 as the snap threshold from ordinal fixed-point structure alone — ω-tower iteration, the fixed-point property of $\alpha \mapsto \omega$^α, and Cantor normal form encodings into $\mathbb{Z}_2$. Remark R-L.1 in the formal document notes a structural alignment with an independent result from proof theory.

Gentzen (1936) proved that ε_0 is the proof-theoretic ordinal of Peano Arithmetic: PA can prove transfinite induction up to any ordinal strictly below ε_0, but not for ε_0 itself. ε_0 is the exact ordinal boundary of PA's own proof machinery — the furthest point its provability reaches before requiring a strictly stronger system.

ZP-L and Gentzen arrive at ε_0 from entirely separate starting points. ZP-L starts from ordinal fixed-point structure: ε_0 is where ω-tower self-iteration closes on itself. Gentzen starts from proof theory: ε_0 is where PA's provability tower runs out. Neither derivation references the other's domain. Both locate the same boundary.

The convergence is notable. ZP-L's fixed-point derivation and Gentzen's proof-theoretic analysis were conducted in separate domains with separate machinery. That both locate ε_0 is a structural observation, not an argument for either result. Both stand or fall on their own proofs.

ZP does not prove any part of Gödel's incompleteness theorems, and does not reprove Gentzen's ordinal analysis of PA. The connection is a structural observation: two independent formal derivations arrive at the same ordinal boundary. Gentzen (1936) established that ε_0 is the specific ordinal where PA's proof-theoretic strength is exhausted. ZP-L established ε_0 as the snap threshold from ordinal fixed-point structure. No formal equivalence between the two derivations is claimed.

A Note on Classical.choice

Every ZP-L theorem carries the axiom footprint [propext, Classical.choice, Quot.sound]. This is the standard footprint of Lean 4 + Mathlib for any theorem that uses ordinal theory, p-adic analysis, or computability. It is not a novel ZP-L commitment.

What is notable is where Classical.choice appears. Across the four mathematical settings of the ZP framework — topology (ZP-B), information theory (ZP-C), set theory and computation (ZP-J/K), and ordinal theory (ZP-L) — the same axiom appears at the same structural step: the non-constructive diagonal. ZP-L §I documents this convergence. Whether Classical.choice is structurally forced by the ZP framework — rather than merely inherited from Mathlib infrastructure — remains an open question.

Key Result — ZP-L: The Ordinal Snap Threshold epsilon_zero_snap_canonical (ZPL.lean §VII): The canonical map $\phi(\alpha) = c_0$ if $\alpha < \varepsilon_0$, else c_1 satisfies all five conditions simultaneously — monotone, tower-aligned (c_0 on every fundamental sequence stage), fixed-point-respecting (c_1 on every fixed point of ω^·), snapping at ε_0, and ε_0 minimal — with no free hypotheses. snap_zp2_correspondence co-proves that the same tower witnesses both the ordinal approach to ε_0 and the 2-adic convergence to $0 = \bot$. All theorems proved without sorry. ZPM formalizes the type bridge between the ordinal and machine-phase encodings. ✓