THE ZERO PARADOX

ZP-L: Incomputability Convergence

Version 1.0 | May 2026

v1.0: Initial release. All theorems §I–§VII proved sorry-free in Lean 4. Axiom footprint: [propext, Classical.choice, Quot.sound] throughout.

ZP-L establishes four results connecting the formal axioms of the ZP framework to standard results in ordinal theory and computability. First, Classical.choice appears at the non-constructive diagonal step in each of the four mathematical settings of the ZP framework — topology, information theory, set theory, and computation. Second, Roger's fixed-point theorem (Kleene's second recursion theorem) is formalized as a wrapper, formalizing the computational fixed-point structure. Third, the ordinal ε_0 is fully characterized as the first fixed point of $\alpha \mapsto \omega$^α and the limit of the tower ω, ω^ω, ω^ω^ω, ... Fourth, ordinals below ε_0 encode into $\mathbb{Z}_2$ via their Cantor normal form, and as the tower stages approach ε_0, their 2-adic encodings converge to 0 = ⊥.

The central result (§VII) is the canonical snap map: $\phi\ \alpha = \text{if } \alpha < \varepsilon_0 \text{ then } c_0 \text{ else } c_1$ simultaneously satisfies all five conditions — monotone, tower-aligned, fixed-point-respecting, snapping at ε_0, and ε_0 minimal. All conditions are verified without free hypotheses for this witness. 24 theorems proved, zero sorry, axiom footprint [propext, Classical.choice, Quot.sound] throughout.

Section I: Axiom Footprint Convergence

Non-constructibility appears in four mathematical settings across the ZP framework. Classical.choice is required at each diagonal step in these proofs — a constructive alternative was not found in any of the four settings.

Layer	Formal Language	Expression of non-constructibility
ZPB	Topology	C3: no continuous path ⊥ → x ≠ ⊥
ZPC	Information Theory	L-INF: infinite surprisal at ⊥
ZPJ/K	Set Theory + Computation	bot_self_mem (AFA); botCode (Kleene)
ZPI	Algorithmic IT	K(S▯|n)/|S▯| → 1; K uncomputable

Remark: Why K is Absent from Lean

Kolmogorov complexity K is not computed in Lean in this framework. Its existence as a total function requires Classical.choice — exactly the axiom Nat.Partrec.Code.fixed_point$_2$ already uses in ZP-K. The AFA/Kleene route reaches the same fixed-point structure via a provable path without requiring K to be explicitly computed.

Axiom footprint evidence — the following ZP-K theorems all carry [propext, Classical.choice, Quot.sound]:

Remark: Why K is Absent from Lean
ZPK.t_comp (T-COMP four-way equivalence)
ZPK.da1_paths_unified
ZPK.isComputationalQuine_undecidable
ZPK.infinite_quine_family
The Classical.choice entry is the computational expression of the diagonal. ZP-L inherits this footprint throughout.

Section II: Roger Fixed-Point Stability

Roger's fixed-point theorem (also known as Kleene's second recursion theorem) states that any computable transformation of a code has a behavioral fixed point: a code c such that running f(c) and running c produce the same partial function. For any computable transformation, at least one fixed-point code exists.

Theorem: roger_fixed_point_stability (ZPL.lean §II)
For any computable f : Code → Code,
∃ c : Code, eval (f c) = eval c
Proof: wrapper around ZPK.roger_fixed_point_exists.
Lean purity: [propext, Classical.choice, Quot.sound]. ✓
Note: this is an existential result. The specific code c is not constructively produced; Classical.choice selects it. This is the same non-constructive step that appears across all ZP layers (§I).

Section III: The Ordinal ε_0

The ordinal ε_0 is the smallest fixed point of the map α ↦ ω^α. It is the supremum of the tower 0, 1, ω, ω^ω, ω^ω^ω, ... — each stage is strictly below ε_0, and ε_0 is not reached by any finite iteration. This section is entirely within Lean scope via Mathlib's ordinal machinery.

Definitions (ZPL.lean §III)
epsilonZero : Ordinal := Ordinal.epsilon 0 (= nfp (ω^·) 0 = veblen 1 0)
fundamentalSeq : ℕ → Ordinal := fun n ↦ (α ↦ ω^α)^[n] 0
Explicit stages:
fundamentalSeq 0 = 0
fundamentalSeq 1 = 1

Definitions (ZPL.lean §III)

fundamentalSeq 2 = ω

fundamentalSeq 3 = ω^ω

fundamentalSeq (n+1) = ω^(fundamentalSeq n)

Theorem: epsilonZero_fixedPoint

ω^$\varepsilon_0 = \varepsilon_0$

Proof: Ordinal.omega0_opow_epsilon 0 (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_eq_nfp

ε_0 = nfp (ω^·) 0

Proof: Ordinal.epsilon_zero_eq_nfp (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_eq_iSup

ε_0 = ⨆ n : ℕ, fundamentalSeq n

Proof: iSup_iterate_eq_nfp (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_tower_lt

∀ n : ℕ, fundamentalSeq n < ε_0

Every finite stage of the tower is strictly below ε_0.

Proof: Ordinal.iterate_omega0_opow_lt_epsilon_zero (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_le_fixedPoint

∀ b : Ordinal, ω^b = b → $\varepsilon_0 \le$ b

ε_0 is the least fixed point of ω^· above 0.

Proof: Ordinal.epsilon_zero_le_of_omega0_opow_le (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: fundamentalSeq_strictMono

$\forall$ n : ℕ, fundamentalSeq n < fundamentalSeq (n + 1)

The tower is strictly monotone.

Proof: by induction; successor step uses isNormal_opow.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Remark R-L.1: Proof-Theoretic Alignment

Gentzen's theorem (1936) establishes that ε_0 is the proof-theoretic ordinal of Peano Arithmetic: PA can prove transfinite induction for any ordinal strictly below ε_0, but not for ε_0 itself. This is not claimed or proved here.

ZP-L derives ε_0 as the snap threshold from ordinal fixed-point structure, independently of proof theory. Both derivations locate the same boundary: the ordinal where ω-tower self-iteration becomes self-limiting. No claim is made that this alignment is more than a structural observation.

Remark R-L.2: Surreal Numbers

Every ordinal is a surreal number (Conway, 1976), so ε_0 lives in the surreal number field No. No satisfies the axioms of a real-closed field, and by Tarski's completeness theorem for the theory of real-closed fields, every first-order sentence in the language of ordered fields that holds in ℝ also holds in No, and vice versa.

ZP-F (The Counterexamples) proves that the Binary Snap — the forced transition from ⊥ to the first non-null ordinal threshold (ε_0, established in §V–§VII above) — cannot occur in any linearly ordered field. This result applies directly to No considered as a linearly ordered field.

The surreals therefore contain ε_0 as an ordinal while simultaneously satisfying the density condition that blocks the Binary Snap in their ordered field structure. Both structures coexist in No: ε_0 is present as an ordinal, and the field density that blocks the snap is present in the field structure. The two results apply to different structural aspects of No.

Remark R-L.3: Hyperreals and Łoś's Theorem

The hyperreals $^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/U$ (ultrafilter U on ℕ) satisfy a transfer result: by Łoś's theorem, every first-order sentence true in ℝ holds in *ℝ. Łoś's theorem applies to the full first-order theory, so any first-order property of ℝ — including field density — transfers.

Field density transfers: between any two hyperreals there is another. The density condition that blocks the Binary Snap in ℝ (proved in ZP-F) therefore holds in *ℝ as well.

The non-standard naturals *ℕ contain infinite elements, but every infinite $H \in {}^*\mathbb{N}$ has a predecessor $H-1$ in *ℕ. ε_0 is a limit ordinal: it has no predecessor and is the supremum of the tower below it. *ℕ cannot host the Binary Snap not only because of field density, but because its infinite elements are successor-like rather than limit-like. The snap requires genuine limit ordinal structure.

The monad of 0 in *ℝ — the external set $\{\varepsilon : |\varepsilon| < 1/n \text{ for all standard } n\}$ — is external to the ultrapower. Between any two distinct elements of the monad there is another (their arithmetic mean in *ℝ), so no discrete jump at 0 is possible in *ℝ.

Remark R-L.3: Hyperreals and Łoś's Theorem

Field density is an internal property of the ordered field structure of $^*\mathbb{R}$ and transfers by Łoś's theorem. Limit ordinal structure is set-theoretic and is not preserved by the ultrapower construction: $^*\mathbb{N}$ contains only successor-like infinite elements, not limit-like ones. The two results — transfer of density, failure of limit-ordinal structure — come from different theorems.

Section IV: Cantor Normal Form Bridge

Every ordinal below ε_0 has a unique Cantor normal form — a finite sum $a_1 \cdot \omega\hat{}e_1 + a_2 \cdot \omega\hat{}e_2 + \ldots$ with strictly decreasing exponents $e_1 > e_2 > \ldots$ and each coefficient a nonzero natural number (a nonzero ordinal strictly below ω). In Lean: NONote (Mathlib.SetTheory.Ordinal.Notation). The encoding cnfToZp2 maps each such ordinal to $\mathbb{Z}_2$ via structural recursion on the Cantor normal form.

Definition: cnfToZp2 (ZPL.lean §IV)

cnfToZp2 : NONote → $\mathbb{Z}_2$

Base: cnfToZp2(0) = 0

Recursive: cnfToZp2(ω^e · n + a) = 2^(v_2(cnfToZp2(e)) + 1) · n + cnfToZp2(a) [n : $\mathbb{N}^+$]

where v_2 denotes the 2-adic valuation.

Valuation of the n-th tower stage:

cnfToZp2(towerNONote 0) = 0 (Lean: PadicInt.valuation 0 = 0; standard: $v_2(0) = +\infty$)

cnfToZp2(towerNONote 1) = 2 (valuation 1)

cnfToZp2(towerNONote 2) = 4 (valuation 2)

cnfToZp2(towerNONote n) = 2^n (valuation n)

Theorem: cnfToZp2_tower_valuation

∀ n : ℕ, (cnfToZp2 (towerNONote n)).valuation = n

The 2-adic valuation of the n-th tower stage encoding equals n.

Proof: by induction; successor step uses PadicInt.valuation_pow.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: cnfToZp2_valuation_unbounded

∀ k : ℕ, ∃ α : NONote, k ≤ (cnfToZp2 α).valuation

The 2-adic valuation is unbounded across ordinals below ε_0.

Theorem: cnfToZp2_valuation_unbounded

Proof: towerNONote k witnesses the bound.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: tower_converges_to_zero

Filter.Tendsto (fun n ↦ cnfToZp2 (towerNONote n)) Filter.atTop (nhds 0)

The tower encodings converge to $0 = \bot$ in $\mathbb{Z}_2$.

Proof: each stage cnfToZp2(towerNONote (n+1)) = 2^(n+1) in $\mathbb{Z}_2$, so its 2-adic norm is ‖2‖^(n+1) = (1/2)^(n+1) → 0. Uses Metric.tendsto_atTop and exists_pow_lt_of_lt_one.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section V: Ordinal Tower Limit and Snap Threshold

This section connects the ordinal tower to the ZPE machine-phase snap. The cofinality theorem shows that the fundamental sequence approaches ε_0 from below. The lower-bound theorem shows that any monotone tower-aligned map must assign c_0 to all pre-ε_0 ordinals. A canonical witness snapping exactly at ε_0 is exhibited.

What this does NOT claim (§V)

Gentzen's theorem: that ε_0 is the proof-theoretic ordinal of PA (not claimed — see Remark R-L.1).

Any statement about formal provability in PA.

That ε_0 is the UNIQUE minimal snap boundary: snap_threshold_is_epsilon_zero shows no ordinal below ε_0 works (for maps satisfying the stated hypotheses), but does not rule out maps that snap at some ordinal strictly above ε_0.

That the snap threshold result applies to all maps Ordinal → MachinePhase regardless of the monotonicity and tower-alignment hypotheses.

Theorem: fundamentalSeq_cofinal

$\forall \alpha$: Ordinal, $\alpha < \varepsilon_0 \to \exists$ n : $\mathbb{N}$, $\alpha <$ fundamentalSeq n

The fundamental sequence is cofinal in ε_0: for any ordinal below ε_0, some tower stage exceeds it.

Proof: ε_0 = nfp (ω^·) 0 (epsilonZero_eq_nfp), and lt_nfp_iff gives a < nfp f b ↔ ∃ n, a < f^[n] b.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: snap_threshold_is_epsilon_zero

For any ϕ : Ordinal → MachinePhase satisfying:

Theorem: snap_threshold_is_epsilon_zero

(a) hmono: $\forall\ \alpha \le \beta$, join $(\phi\ \alpha)\ (\phi\ \beta) = \phi\ \beta$ (order-non-decreasing in the ZPSemilattice sense: $\phi\ \alpha \le \phi\ \beta$)

(b) h0: $\forall\ n : \mathbb{N}$, ϕ (fundamentalSeq n) $= c_0$

we have: $\forall\ \alpha < \varepsilon_0$, $\phi\ \alpha = c_0$

This is a lower bound: no ordinal below ε_0 is a snap point for maps satisfying (a) and (b). "Minimal" not "unique."

Proof: cofinality gives a stage above α; monotonicity chains $\phi\ \alpha \le \phi(\text{stage}) = c_0$; $c_0 = \bot$ forces $\phi\ \alpha = c_0$.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: c1_epsilon_zero_identification

$\exists\ \phi$: Ordinal → MachinePhase,

$(\forall\ n : \mathbb{N},\ \phi\ (\text{fundamentalSeq } n) = c_0) \wedge \phi\ \varepsilon_0 = c_1$

Witness: $\phi\ \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1.

One specific map sends all tower stages to c_0 and ε_0 to c_1.

The stronger structural claim — an order-preserving morphism Ordinal →□ MachinePhase compatible with the CNF → $\mathbb{Z}_2$ encoding — remains outside Lean scope: no type bridge between Ordinal and MachinePhase is defined in this library.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section VI: Kleene-Ordinal Fixed-Point Bridge

The ordinal fixed-point structure ($\varepsilon_0 = \text{nfp}\ (\omega\text{^}\cdot)\ 0$, $\omega\text{^}\varepsilon_0 = \varepsilon_0$) and the computational fixed-point structure (Kleene's recursion theorem, roger_fixed_point_stability) both require Classical.choice at their non-constructive step — parallel structure, not a proved isomorphism. The hypothesis hfp encodes that ordinal fixed points of $\omega\text{^}\cdot$ map to the snap state c_1. Under this hypothesis plus monotonicity and tower alignment, ε_0 is the minimal snap threshold.

Theorem: snap_exactly_at_epsilon_zero

For any ϕ : Ordinal → MachinePhase satisfying:

(a) hmono: order-non-decreasing (join $(\phi\ \alpha)\ (\phi\ \beta) = \phi\ \beta$ for $\alpha \le \beta$)

(b) h0: every tower stage maps to c_0

(c) hfp: every fixed point of $\omega\text{^}\cdot$ maps to c_1

we have: $\phi\ \varepsilon_0 = c_1$ AND $\forall\ \alpha,\ \phi\ \alpha = c_1 \to \varepsilon_0 \le \alpha$

Theorem: snap_exactly_at_epsilon_zero

ε_0 is the minimal snap threshold: ϕ assigns c_1 first at ε_0.

Note: "minimal" not "unique." A map satisfying these hypotheses could also assign c_1 to ordinals above ε_0; what is ruled out is any snap strictly before ε_0.

Proof: upper bound from hfp + epsilonZero_fixedPoint; lower bound from snap_threshold_is_epsilon_zero.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: kleene_ordinal_snap_bridge

$\exists\, \phi$: Ordinal → MachinePhase,

($\forall$ n : ℕ, ϕ (fundamentalSeq n) = c_0) ∧

($\forall\, \alpha$, ω^α = α → $\phi\, \alpha = c_1$) ∧

$\phi\, \varepsilon_0 = c_1$ ∧

$\forall\, \alpha, \phi\, \alpha = c_1 \to \varepsilon_0 \le \alpha$

Witness: $\phi\, \alpha$ = if $\alpha < \varepsilon_0$ then c_0 else c_1.

Note: the theorem name refers to the informal §VI conceptual parallel between Kleene recursion fixed points and ordinal fixed points of ω^·. The theorem itself is purely ordinal — no Code or eval appears.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section VII: Canonical Snap Map — Full Closure

The canonical threshold map $\phi\, \alpha$ = if $\alpha < \varepsilon_0$ then c_0 else c_1 satisfies all three conditions of snap_exactly_at_epsilon_zero (hmono, h0, hfp). For this specific map, the snap identification is unconditional: all five conditions are simultaneously verified without free hypotheses.

Theorem: snap_map_mono

$\forall\, \alpha\, \beta$: Ordinal, $\alpha \le \beta \to$

join (if $\alpha < \varepsilon_0$ then c_0 else c_1)

(if $\beta < \varepsilon_0$ then c_0 else c_1) =

if $\beta < \varepsilon_0$ then c_0 else c_1

The canonical map is order-non-decreasing.

Proof: four cases (α/β vs ε_0). The case $\alpha \ge \varepsilon_0$ with $\beta < \varepsilon_0$ is impossible by $\alpha \le \beta$. Each live case closes by rfl from the MachinePhase join definition.

Theorem: snap_map_mono

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilon_zero_snap_canonical

$\exists\, \phi$: Ordinal → MachinePhase,

($\forall\, \alpha\, \beta, \alpha \le \beta \to$ join ($\phi\, \alpha$) ($\phi\, \beta$) $= \phi\, \beta$) ∧ [hmono]

($\forall$ n : ℕ, ϕ (fundamentalSeq n) $= c_0$) ∧ [h0]

($\forall\, \alpha$, ω^α = α → $\phi\, \alpha = c_1$) ∧ [hfp]

$\phi\, \varepsilon_0 = c_1$ ∧ [snap]

$\forall\, \alpha, \phi\, \alpha = c_1 \to \varepsilon_0 \le \alpha$ [minimality]

All five conditions verified for the explicit witness $\phi\, \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1, with no free hypotheses.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: snap_zp2_correspondence

Four independent facts about the same tower sequence:

(i) $\forall$ n : ℕ, fundamentalSeq n $< \varepsilon_0$

(ii) $\forall$ n : ℕ, ϕ (fundamentalSeq n) $= c_0$ where ϕ is the canonical map

(iii) Filter.Tendsto (fun n ↦ cnfToZp2 (towerNONote n)) Filter.atTop (nhds 0)

(iv) $\phi\, \varepsilon_0 = c_1$

The same tower sequence witnesses both the ordinal approach to ε_0 and the 2-adic approach to 0. The full structural identification ($\varepsilon_0 \leftrightarrow \bot$ via a type bridge) remains outside Lean scope — see §V.

Proof: ⟨epsilonZero_tower_lt, fun n ↦ if_pos ..., tower_converges_to_zero, if_neg (lt_irrefl ε_0)⟩

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Remaining Formal Gap

The Remaining Gap

The identification of ZPE's MachinePhase element c_1 with the ordinal ε_0 via a formal type bridge remains outside Lean scope.

The Remaining Gap

What is proved: a canonical map Ordinal → MachinePhase assigns c_1 exactly at ε_0 and nowhere earlier. What is not proved: a canonical ZPSemilattice morphism MachinePhase → $\mathbb{Z}_2$ that would connect ZPE's $\bot = c_0$ to ZPB's $\bot = 0$ formally.

Such a bridge would require:

(1) snapEmbed : MachinePhase → $\mathbb{Z}_2$ mapping $c_0 \mapsto 0$, $c_1 \mapsto 1$

(2) proof that snapEmbed is join-preserving

(3) a bridge theorem deriving hfp from tower_converges_to_zero via snapEmbed

This would make hfp a theorem rather than a hypothesis in snap_exactly_at_epsilon_zero. The canonical witness (epsilon_zero_snap_canonical) satisfies all five conditions without this bridge; the bridge would close the gap between the two formal instances of ⊥ across ZPE and ZPB.

Theorem Summary

Theorem	Section	Status
roger_fixed_point_stability	§II	Proved ✓
epsilonZero_fixedPoint	§III	Proved ✓
epsilonZero_eq_nfp	§III	Proved ✓
epsilonZero_eq_iSup	§III	Proved ✓
epsilonZero_tower_lt	§III	Proved ✓
epsilonZero_le_fixedPoint	§III	Proved ✓
fundamentalSeq_strictMono	§III	Proved ✓
tower_stage_zero / _one / _two	§III	Proved ✓
towerNONote_repr	§IV	Proved ✓
cnfToZp2_tower_valuation	§IV	Proved ✓
cnfToZp2_valuation_unbounded	§IV	Proved ✓
fundamentalSeq_zp2_converges	§IV	Proved ✓
tower_converges_to_zero	§IV	Proved ✓
zpe_snap_ordinal_correspondence	§V	Proved ✓
epsilonZero_tower_bound	§V	Proved ✓
c1_epsilon_zero_identification	§V	Proved ✓

Theorem	Section	Status
fundamentalSeq_cofinal	§V	Proved ✓
snap_threshold_is_epsilon_zero	§V	Proved ✓
snap_exactly_at_epsilon_zero	§VI	Proved ✓
kleene_ordinal_snap_bridge	§VI	Proved ✓
snap_map_mono	§VII	Proved ✓
epsilon_zero_snap_canonical	§VII	Proved ✓
snap_zp2_correspondence	§VII	Proved ✓

Axiom Purity
All 23 theorems carry axiom footprint: [propext, Classical.choice, Quot.sound].
These are standard Mathlib infrastructure axioms (ordinal theory, p-adic analysis, computability). They are not ZP-L commitments.
Classical.choice is load-bearing: it is the formal non-constructivity appearing at the diagonal step in each ZP layer (§I). Its presence is expected and documented, not incidental.
Zero sorry in ZPL.lean. Verified: lake build, May 2026.