THE ZERO PARADOX

ZP-N: The Constructive Snap

the ε_0 snap from below, choice-free on ordinal notations

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The choice-free constructive companion to ZP-L. The snap-from-below is rebuilt syntactically on ordinal notations (ONote), never touching Mathlib's choice-saturated Ordinal type; the three snap results are choice-free — [propext] only — in contrast to every ε_0 result in ZP-L, which carries Classical.choice. Proved sorry-free in Lean 4 (ConstructiveOrdinals.lean).

ZP-L derived the snap at ε_0, but every ε_0 result in its Lean development carries Classical.choice — inherited from Mathlib's Ordinal type, which is choice-saturated. ZP-N asks whether that choice is intrinsic to the snap or merely representational. The answer is representational. The snap's downward structure — the ω-tower climbs without bound, and no ordinal notation is a fixed point of $x \mapsto \omega^x$ — is genuinely constructive, provable with no Axiom of Choice.

The move is to work at the level of ordinal notations (ONote, Cantor-normal-form terms), whose comparison ONote.cmp is choice-free (propext-only) and never passes through repr into the semantic Ordinal type. ε_0 is not itself a notation: the notations name exactly the ordinals below ε_0, and ε_0 is their limit — the fixed point the notation system cannot reach. So the snap threshold is characterised from below: the tower ascends, and the fixed point (ε_0) is precisely what lies beyond every notation. This layer adds no axiom; it isolates where choice actually enters the ordinal snap.

Section I: The Syntactic Substrate

The Cantor normal form gives each ordinal below ε_0 a finite syntactic name. At the notation level, ω^x is simply the term oadd x 1 0 (leading exponent x, coefficient 1, empty remainder) — no general ordinal exponentiation is needed. The ω-tower is built by iterating it. Comparison of notations is the purely syntactic ONote.cmp, which returns lt / eq / gt by structural recursion and never evaluates repr, so it is choice-free.

Definitions (ConstructiveOrdinals.lean)
omegaPow x = oadd x 1 0 — the notation for ω^x; purely syntactic and computable.
tower 0 = 0, tower (n+1) = omegaPow (tower n) — the ω-tower $(\omega^{\cdot})^{[n]}\ 0$, constructive.
ONote.cmp — the syntactic three-way comparison on notations (lt / eq / gt); choice-free (it does not touch repr / Ordinal).

Section II: The Choice-Free Snap from Below

Lemma: exp_lt_term (ConstructiveOrdinals.lean)

For every notation e, coefficient n, and remainder a: cmp e (oadd e n a) = lt.

An exponent is strictly below its own term. Proof: structural induction on the exponent, pure syntax — no repr, no Ordinal.

Lean purity: [propext] only — choice-free, and free even of Quot.sound. ✓

Theorem: omegaPow_no_fixedpoint (ConstructiveOrdinals.lean)

For every notation x: cmp x (omegaPow x) = lt.

No ordinal notation is a fixed point of $x \mapsto \omega^x$: every notation is strictly below its own ω-power, in the choice-free syntactic comparison (holds for all notations, well-formed or not). This is the constructive shadow of "ε_0 is the least fixed point of $x \mapsto \omega^x$, lying beyond every notation."

Lean purity: [propext] only — choice-free. ✓

Theorem: tower_strictMono (ConstructiveOrdinals.lean)

For every n: cmp (tower n) (tower (n+1)) = lt.

The ω-tower is strictly increasing: the snap stages climb without bound below ε_0. Immediate from omegaPow_no_fixedpoint applied at tower n.

Lean purity: [propext] only — choice-free. ✓

The snap from below is choice-free. All three results depend on [propext] alone — not Classical.choice, and not even Quot.sound. The ascent of the ω-tower and the absence of any notation fixed point are proved by pure structural recursion on the syntax. This is the sharpest form of the framework's choice-free-core pattern (see ZeroParadox/AxiomProfile.lean): the snap's downward structure needs nothing beyond propositional extensionality.

Section III: The Finding and Its Fences

The point of ZP-N is the contrast with ZP-L. There, every ε_0 result carries Classical.choice; here, the same downward structure is proved with [propext] only. The difference is entirely the substrate: ZP-L works in Mathlib's Ordinal type, which is choice-saturated, while ZP-N works in ONote, which is not. So the Classical.choice in ZP-L's snap is representational, not intrinsic: it is inherited from the semantic representation, and vanishes once the same fact is stated syntactically. Choice enters the ordinal snap only at the syntax→semantics bridge, not in the snap itself.

Remark: even well-formedness inherits choice (the bridge, made visible)

tower_NF — the statement that each tower stage is in normal form — DOES carry Classical.choice ([propext, Classical.choice, Quot.sound]), because Mathlib's NF predicate is defined through repr into Ordinal. The snap facts (Section II) do not depend on NF, so they stay choice-free; but the fact that even "this notation is well-formed" inherits choice pins the location precisely: choice lives at the syntax→semantics bridge, exactly where ZP-L crosses it and ZP-N does not.

Scope fence: the snap from below, not minimality

ZP-N proves the snap from below: ε_0 is the fixed point unreachable from within the notation system (no notation is a fixed point of $x \mapsto \omega^x$; the tower climbs without bound). The matching minimality — that ε_0 is the LEAST fixed point of $x \mapsto \omega^x$ — is a separate, harder target: it quantifies over the limit, which no notation names, so it cannot be stated purely syntactically on ONote. That direction remains open; what is proved here is the from-below half, choice-free.

Theorem Summary

Result	Lean source (full path)	Axioms
exp_lt_term	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only
omegaPow_no_fixedpoint	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only
tower_strictMono	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only
tower_NF	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	Classical.choice

Axiom Purity

The snap from below (exp_lt_term, omegaPow_no_fixedpoint, tower_strictMono): [propext] only — choice-free, and free even of Quot.sound. Proved by structural recursion on ONote; repr / Ordinal are never touched.

Well-formedness (tower_NF): [propext, Classical.choice, Quot.sound] — Classical.choice inherited from Mathlib's NF / repr, which passes through the Ordinal type.

This localises the Classical.choice in ZP-L's ε_0 results to the syntax→semantics bridge: the snap itself is constructive. Zero sorry. Verified: lake build, July 2026.

End of ZP-N | The Constructive Snap | the snap from below on ordinal notations, choice-free ([propext] only): exp_lt_term, omegaPow_no_fixedpoint, tower_strictMono | ZP-L's ε_0 choice is representational, not intrinsic | tower_NF carries choice, at the syntax→semantics bridge | minimality (ε_0 the least fixed point) open.