

THE ZERO PARADOX

ZP-P: The Fixed-Point Fork

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Synthesis layer. The abstract fork schema is proved sorry-free and choice-free in Lean 4 (`FixedPointFork.lean`); the number-system instance via Ostrowski (`Ostrowski.lean`) and the categorical-parent instance via `QPF.Fix / QPF.Cofix` (`Coalgebra.lean`) are Lean-witnessed; the set-theory and computation instances are referenced to ZP-J and ZP-K. Generalizes the ZFC+Foundation / ZFC+AFA orthogonal-contact-point claim.

A self-referential operator over a complete lattice has a least fixed point and a greatest fixed point (Knaster–Tarski). ZP-P records a single elementary fact and develops its consequences across the framework: these two fixed points coincide — the fork collapses to one point — exactly when the operator has a unique fixed point. Read intuitively, the least fixed point is the inductive (well-founded) closure and the greatest is the coinductive (non-well-founded) closure; the collapse point is where the two closures meet. In the Zero Paradox that point is the diagonal fixed point $\perp$, the self-containing bottom.

This layer is a synthesis layer, not a foundational one: it adds no axiom or primitive. It names a pattern that several existing layers already realise, and it is organised in three tiers held to three different standards of proof. Tier 1 (Section I) is the abstract schema, proved in Lean over a complete lattice. Tier 2 (Section II) is the instance catalogue: each framework forks at its own contact point, and the instances are theorems in their home layers. Tier 3 (Section III) is the unification — that the contact points are faces of one object — which is a fenced conjecture, never a formal claim. The fences in Section III are load-bearing: keeping the tiers separate is what makes the formal content honest.

Section I: The Fork Schema (Tier 1)

Over a complete lattice, a monotone self-map f has a least fixed point ($\text{lfp } f$) and a greatest fixed point ($\text{gfp } f$), and $\text{lfp } f \leq \text{gfp } f$ always. The width between them is the fork. The schema theorem states that the fork collapses to a single point precisely when f has a unique fixed point — and in that case the common value is that fixed point. The proofs rest only on Mathlib's order-theoretic fixed-point API (`OrderHom.lfp / OrderHom.gfp`); they make no claim about the categorical inductive/coinductive reading, which is offered as analogy only.

Definition: the fork (`FixedPointFork.lean`)

For a complete lattice α and a monotone self-map $f : \alpha \rightarrow \alpha$:

$\text{lfp } f$ = the least fixed point of f (`Mathlib OrderHom.lfp`)

$\text{gfp } f$ = the greatest fixed point of f (`Mathlib OrderHom.gfp`)

The fork is the ordered pair $(\text{lfp } f, \text{gfp } f)$, always satisfying $\text{lfp } f \leq \text{gfp } f$.

Collapse = $\text{lfp } f = \text{gfp } f$: the two ends meet at a single contact point.

Proposition: fork_le (FixedPointFork.lean)

$\text{lfp } f \leq \text{gfp } f$

The inductive closure never exceeds the coinductive closure: the fork has non-negative width. (Mathlib OrderHom.lfp_le_gfp.)

Lean purity: [propext, Quot.sound] — choice-free. ✓

Lemma: collapse_of_unique (FixedPointFork.lean)

If x is the unique fixed point of f , then $\text{lfp } f = x$ and $\text{gfp } f = x$.

Both ends of the fork land on the sole fixed point.

Proof: $\text{lfp } f$ and $\text{gfp } f$ are fixed points (map_lfp , map_gfp); uniqueness sends each to x .

Lean purity: [propext, Quot.sound] — choice-free. ✓

Lemma: unique_of_collapse (FixedPointFork.lean)

If $\text{lfp } f = \text{gfp } f$, then every fixed point of f equals that common value.

Every fixed point lies between $\text{lfp } f$ and $\text{gfp } f$, so collapse forces uniqueness.

Proof: lfp_le_fixed and le_gfp bracket any fixed point y ; antisymmetry closes it.

Lean purity: [propext, Quot.sound] — choice-free. ✓

Theorem: fork_collapse_iff (FixedPointFork.lean) — the schema spine

$\text{lfp } f = \text{gfp } f \leftrightarrow \exists! x, f x = x$

The fork collapses to a single contact point if and only if f has a unique fixed point.

This is the abstract form of "the framework forks, and the diagonal fixed point is where the two closures meet."

Proof: $\text{collapse_of_unique}$ and $\text{unique_of_collapse}$, both directions.

Lean purity: [propext, Quot.sound] — choice-free. ✓

The fork spine is choice-free. All four results depend only on [propext, Quot.sound] — propositional extensionality and quotient soundness, the benign kernel axioms used throughout Mathlib. None depends on the Axiom of Choice. This is consistent with the choice-free core (see AxiomProfile.lean): the conceptual skeleton needs no choice; choice enters only in the analytic realisations (Section II).

Section II: The Instance Catalogue (Tier 2)

The same fork appears across frameworks. ZP-P does not re-prove each instance — each is a theorem in its home layer — it records the catalogue and pins the number-system instance to a checkable Lean witness. The original ZFC+Foundation / ZFC+AFA orthogonal-contact-point claim is the set-theory entry: in the standard category-theoretic characterisation (Aczel; Lambek), Foundation is the initial algebra (μ) and AFA the final coalgebra (ν) of the bounded powerset functor, and the Quine atom $\perp = \{\perp\}$ is exactly an element present in ν but not in μ .

Domain	The two closures fork into	Contact point / home layer
Set theory	Foundation (μ , well-founded) vs AFA (ν , non-well-founded)	Quine atom $\perp = \{\perp\}$; ZP-J / ZP-K
Number systems	Archimedean $\mathbb{R}$ vs non-Archimedean $\mathbb{Q}_p$	0 (snap fails in $\mathbb{R}$, holds in $\mathbb{Q}_2$); ZP-B / ZP-F
Computation	total (well-founded) vs partial (non-terminating)	the Kleene quine / self-application; ZP-K
Category theory	initial algebra (W-types) vs final coalgebra (M-types)	QPF.Fix vs QPF.Cofix; Coalgebra.lean

The number-system instance is the one with a genuine classification theorem behind it. Ostrowski’s theorem says the nontrivial absolute values on $\mathbb{Q}$ are exactly the real (Archimedean) one and the p-adic (non-Archimedean) ones — the complete, pairwise-inequivalent list. The contact point is 0: in the Archimedean completion $\mathbb{R}$ it is approached but never reached (density — the snap fails), while in the 2-adic completion $\mathbb{Q}_2$ it is the unique fixed point of $x \mapsto 2x$ and its valuation diverges ($v_2(0) = \infty$ — the snap holds).

Theorem: completions_exhaustive (Ostrowski.lean)
Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to the real absolute value, or to a p-adic absolute value for a unique prime p.
The two kinds of completion are the complete list. (Ostrowski’s theorem, Mathlib Rat.AbsoluteValue.equiv_real_or_padic.)
Lean purity: [propxt, Classical.choice, Quot.sound] — choice-carrying. ✓

Theorem: real_not_equiv_padic (Ostrowski.lean)
The real absolute value is inequivalent to every p-adic absolute value.
The two kinds of completion are genuinely distinct — never the same metric. This is the orthogonality leg of the number-system fork. (Mathlib Rat.AbsoluteValue.not_real_isEquiv_padic.)
Lean purity: [propxt, Classical.choice, Quot.sound] — choice-carrying. ✓

Remark: choice-free spine, choice-carrying realisation
The fork spine (Section I) is choice-free [propxt, Quot.sound]. The Ostrowski instance carries Classical.choice, inherited from Mathlib’s classical analysis and number theory. This split is the framework’s standing pattern: the conceptual core is choice-free, while the analytic realisations are choice-carrying.

Remark: choice-free spine, choice-carrying realisation

The number-system fork is theorem-backed on its own terms — Ostrowski is a genuine classification theorem, stronger backing than the metatheoretic Foundation/AFA orthogonality. It is NOT claimed to be an instance of the μ/ν (least-vs-greatest fixed point) schema: Ostrowski concerns absolute values, not fixed points of a functor. The thread to the diagonal fixed point runs through the contact point 0 (the unique fixed point of $x \mapsto 2x$ in $\mathbb{Q}_2$), not through the schema theorem.

The categorical instance is the genuine μ/ν case — the parent of the set-theory entry. On a one-shape, one-child polynomial functor (no leaf), the initial algebra (W-type, μ) is empty while the final coalgebra (M-type, ν) is inhabited by the infinite self-referential element: the non-well-founded closure contains a point the well-founded closure lacks — the categorical analog of the Quine atom present in νF but not μF .

Theorem: categorical_fork_strict (Coalgebra.lean)

`IsEmpty (Fix idPF.Obj)  $\wedge$  Nonempty (Cofix idPF.Obj)`

The initial algebra (least fixed point, μ) is empty; the final coalgebra (greatest fixed point, ν) is inhabited. (Mathlib `QPF.Fix` / `QPF.Cofix`.)

Split footprint: `fix_isEmpty` (μ empty) is choice-free [propxet, Quot.sound]; `cofix_nonempty` (ν inhabited) carries `Classical.choice` from the M-type / corecursion machinery — a library artifact, not a necessity: for a polynomial functor the final coalgebra is constructible choice-free (Veltri, FSCD 2021). ✓

Remark: where choice genuinely enters the μ/ν fork

The `Classical.choice` in `cofix_nonempty` is inherited from Mathlib's M-type machinery, not forced by the mathematics: for a polynomial functor the final coalgebra is constructible choice-free (Ahrens–Capriotti–Spadotti; Veltri, FSCD 2021, the coinductive construction).

Choice genuinely enters only for the non-polynomial finite-powerset functor, where the literature pins each presentation: the set-quotient requires the full axiom of choice for finality, while Worrell's $(\omega+\omega)$ -limit requires countable choice together with the lesser limited principle of omniscience (LLPO) — indeed injectivity of the canonical algebra is equivalent to LLPO (Veltri, FSCD 2021).

The remaining two instances are referenced, not re-proved here. The set-theory fork (Foundation vs AFA) has its Lean witness in ZP-J: the Quine atom $\perp = \{\perp\}$ is the unique self-application fixed point (`quine_atom_unique`). The computation fork (total vs partial) has its Lean witness in ZP-K: the Kleene quine. Each is the contact point of its own fork.

Section III: The Unification and Its Fences (Tier 3)

The tempting claim is that the contact points of all the instances — the Quine atom, the 2-adic 0, the Kleene quine, the categorical initial object — are faces of one object, the diagonal fixed point. ZP-P states this as a conjecture and fences it precisely. The fences are not hedging; they mark a genuine boundary, and per-instance forks remain full theorems on either side of them.

Hard fence (permanent): cross-instance identity is not a theorem

The claim "the contact points are one object" is not a theorem and cannot become one. Identity requires a shared type: the 2-adic 0, the Quine atom (a set), the Kleene quine (a code), and a categorical initial object (an object up to isomorphism) are terms of different types in different categories. The proposition " $x = y$ " across them is not false — it is not well-formed. So this is a type boundary, not a missing proof. What is claimed formally is only that each framework forks at its own contact point.

Soft fence (conjecture): not every fork is a μ/ν instance

That every domain fork is an instance of the least-vs-greatest fixed point schema is an open generalisation, not established here. The schema is a theorem in this layer and for canonical functors (W-types vs M-types). The number-system fork (Ostrowski) is the clear case that is theorem-backed yet NOT visibly a μ/ν instance — it is the live caution against overstating the generalisation.

Per-instance forks are theorems and are not hedged. The fences scope only the cross-instance identity (Tier 3) and the universal schema claim. Each individual fork — Foundation/AFA, $\mathbb{R}/\mathbb{Q}_2$, total/partial — stands on its own proof.

Theorem Summary

Result	File / Section	Axioms
fork_le	FixedPointFork.lean §I	choice-free
collapse_of_unique	FixedPointFork.lean §I	choice-free
unique_of_collapse	FixedPointFork.lean §I	choice-free
fork_collapse_iff	FixedPointFork.lean §I	choice-free
completions_exhaustive	Ostrowski.lean §II	Classical.choice
real_not_equiv_padic	Ostrowski.lean §II	Classical.choice
fix_isEmpty	Coalgebra.lean §II	choice-free
cofix_nonempty	Coalgebra.lean §II	Classical.choice

Axiom Purity

Fork spine (FixedPointFork.lean): [propext, Quot.sound] — choice-free. The schema needs no Axiom of Choice.

Number-system instance (Ostrowski.lean): [propext, Classical.choice, Quot.sound] — Classical.choice inherited from Mathlib's classical analysis / number theory (Ostrowski).

Categorical instance (Coalgebra.lean): split — fix_isEmpty (μ empty) is choice-free [propext, Quot.sound]; cofix_nonempty (ν inhabited) carries Classical.choice from the M-type / corecursion machinery.

Core choice-free, realisation choice-carrying — the framework's standing pattern. Zero sorry. Verified: lake build, June 2026.

End of ZP-P | The Fixed-Point Fork | fork_collapse_iff (choice-free spine) | Ostrowski number-system instance |
categorical μ/ν instance (Fix empty / Cofix inhabited) | hard fence: cross-instance identity is a type boundary | soft fence:
not every fork is μ/ν | All FixedPointFork.lean / Ostrowski.lean / Coalgebra.lean theorems verified.