THE ZERO PARADOX

ZP-A: Lattice Algebra

Version 1.1 | April 2026

Supersedes v1.0 | T4 reclassified as Conditional Claim CC-1

This document is self-contained within abstract algebra. No topology, probability, or Hilbert space is imported. Every claim is provable using only the tools of semilattice theory. Cross-framework connections are deferred to ZP-E.

Version 1.1 change: Theorem T4 is reclassified as Conditional Claim CC-1. The v1.0 label "Theorem" was imprecise: the result holds only given the assumption that the state sequence is initialised at the minimum of L. This assumption is not derived from A1–A4 — it is a modelling commitment. The reclassification makes the epistemic status explicit without changing the content or downstream use of the result.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple: $(L, \vee, \bot)$

L is a non-empty set (the carrier set of states). $\vee : L \times L \to L$ is a binary operation called join. $\bot \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom
A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$
A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$
A3 — Idempotency: $x \vee x = x$ for all $x \in L$
A4 — Identity (Additive): $\bot \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes $\bot$ the additive identity of the algebra: the element that contributes nothing to a join and is therefore present in every state as the neutral constituent.

II. The Induced Partial Order

2.1 Definition of ≤

Definition D1 — Lattice Order
For $x, y \in L$, define:

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$x \leq y :\Longleftrightarrow x \vee y = y$

Theorem T1 — ≤ is a Partial Order

Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓

Antisymmetry: if $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓

Transitivity: if $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 ⊥ is the Least Element

Theorem T2 — ⊥ is a Global Minimum under ≤

For all $x \in L$: $\bot \leq x$

Proof: By A4, $\bot \vee x = x$. By D1, this is the definition of $\bot \leq x$. ✓

T2 is the algebraic statement of the foundational claim: ⊥ is not a void that states depart from — it is the minimum element that every state sits above. Since $\bot \leq x$ for all x, and join accumulates from the bottom, ⊥ is algebraically present in every element of L.

III. The Additive Ontology

3.1 No Subtraction Operator

Remark R1 — Join-Semilattice vs. Lattice

A full lattice $(L, \vee, \wedge, \bot, \top)$ includes a meet operator ∧ and a top element ⊤. The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — the removal of informational content from a state. The additive ontology requires that no operation decreases informational content.

3.2 Join is the Only State Transition

Definition D2 — State Transition

A state transition is any function $f: L \to L$ such that $x \leq f(x)$ for all $x \in L$.

Equivalently, $f(x) = x \vee \alpha$ for some $\alpha \in L$.

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence
A state sequence is a function $S: \mathbb{N} \to L$, written $(S_0, S_1, S_2, \ldots)$, such that:
$S_{n+1} = S_n \vee \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Theorem T3 — State Sequences are Monotone
For any state sequence (S_n) satisfying D3: $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$
Proof: By D3, $S_{n+1} = S_n \vee \alpha_n$. By D1, $S_n \leq S_n \vee \alpha_n$ iff $S_n \vee (S_n \vee \alpha_n) = S_n \vee \alpha_n$. By A1, $(S_n \vee S_n) \vee \alpha_n = S_n \vee \alpha_n$. By A3, $S_n \vee S_n = S_n$. Therefore $S_n \vee \alpha_n = S_{n+1}$. ✓
Monotonicity is a theorem, not a postulate. It is derived from A1–A3 via D3.

4.2 The Initial State

Conditional Claim CC-1 — $S_0 = \bot$ (Reclassified from T4 in v1.0)
If the state sequence begins at the minimum of L, then $S_0 = \bot$, and by T2 and T3:
$\bot \leq S_0 \leq S_1 \leq S_2 \leq \ldots$
Corollary: Every state S_n satisfies $\bot \leq S_n$, confirming $\bot$ is a constituent of every state in the sequence.
Status: CONDITIONAL CLAIM — $S_0 = \bot$ is a modelling commitment, not derived from A1–A4. Content unchanged from v1.0; classification corrected.

V. Open Question OQ-A1

Open Question OQ-A1 — Sufficiency of Monotonicity
Is the monotonicity constraint (T3) sufficient to characterise all valid state sequences, or are additional axioms required?
OQ-A1a: Is there algebraic reason to restrict α_n to join-irreducible elements (not expressible as joins of strictly smaller elements)?
OQ-A1b: Does the open-ended semilattice (without top element $\top$) permit unbounded ascending chains?
Status: Open within ZP-A. Closed by ZP-E Theorem T5 (Iterative Forcing Theorem) via AX-B1 from ZP-B.

VI. Boundary Conditions

Export	Status / Receiving Document
$(L, \vee, \bot)$ as join-semilattice	Derived (A1–A4) — ZP-D: algebraic structure of state space

Export	Status / Receiving Document
≤ partial order (D1, T1)	Derived — ZP-D: ordering on states
Monotonicity of state sequences (T3)	Derived from A1–A3 — ZP-D: state layer ordering
⊥ as global minimum (T2, CC-1)	Derived / Conditional — ZP-E: ontological grounding claim
No subtraction / additive ontology (R1)	Structural — ZP-C: no operation may reduce informational content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5

VII. Validation Status

Component	Status / Notes
A1–A4 join-semilattice axioms	Valid — Axioms; self-contained
≤ partial order (D1, T1)	Valid — Derived from A1–A3
⊥ as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1–A3 and D3
CC-1: $S_0 = \bot$	Conditional Claim — modelling commitment; not derived from A1–A4
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5