

THE ZERO PARADOX

ZP-A: Lattice Algebra

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This document is self-contained within abstract algebra. No topology, probability, or Hilbert space is imported. Every claim is provable using only the tools of semilattice theory. Cross-framework connections are deferred to ZP-E.

Illustrated Companion: A paired ZP-A Illustrated Companion document provides concrete examples and visual intuitions for the results in this document. Examples are kept separate from the formal layers to distinguish illustrative material from proofs. The companion is a reading aid; no proof-critical judgements should be drawn from examples alone.

Metatheoretic Declaration — ZF + AFA

This document is stated over ZF + AFA (Zermelo–Fraenkel set theory with Aczel’s Anti-Foundation Axiom). AFA replaces the classical Axiom of Foundation and permits self-containing sets — in particular, sets satisfying $x = \{x\}$.

Scope: This declaration affects only CC-2 (Section V), which asserts $\perp = \{\perp\}$. All algebraic results in Sections I–IV are independent of AFA and hold in standard ZF.

Standard concrete models (power sets ordered by inclusion, real intervals ordered by max, etc.) satisfy A1–A4 but do not satisfy $\perp = \{\perp\}$. This is expected — they are models of the algebraic structure, not instantiations of the ZF + AFA metatheory. The self-containment of \perp is a set-theoretic claim about what \perp is, not an algebraic one.

The Axiom of Choice is not assumed. AFA is forced rather than chosen — see Section V and ZP-E Remark R-AFA for the argument.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple: (L, \vee, \perp)

L is a non-empty set (the carrier set of states). $\vee : L \times L \rightarrow L$ is a binary operation called join. $\perp \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom

A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$

A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$

A3 — Idempotency: $x \vee x = x$ for all $x \in L$

A4 — Identity (Additive): $\perp \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes \perp the additive identity of the algebra: the element that contributes nothing to a join and is therefore present in every state as the neutral constituent.

II. The Induced Partial Order

2.1 Definition of \leq

Definition D1 — Lattice Order

For $x, y \in L$, define the relation \leq by:

$$x \leq y \iff x \vee y = y$$

Proposition T1 — \leq is a Partial Order

Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓

Antisymmetry: if $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓

Transitivity: if $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 \perp is the Least Element

Lemma T2 — \perp is a Global Minimum under \leq

For all $x \in L$: $\perp \leq x$

Proof: By A4, $\perp \vee x = x$. By D1, this is the definition of $\perp \leq x$. ✓

T2 is the algebraic statement of the foundational claim: \perp is not a void that states depart from — it is the minimum element that every state sits above. Since $\perp \leq x$ for all x , and join accumulates from the bottom, \perp is algebraically present in every element of L .

III. The Additive Ontology

3.1 No Subtraction Operator

Remark R1 — Join-Semilattice vs. Lattice

A full lattice $(L, \vee, \wedge, \perp, \top)$ includes a meet operator \wedge and a top element \top . The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — the removal of informational content from a state. The additive ontology requires that no operation decreases informational content.

3.2 Join is the Only State Transition

Definition D2 — State Transition

A state transition is any function $f: L \rightarrow L$ such that $x \leq f(x)$ for all $x \in L$.

Definition D2 — State Transition

Equivalently, for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$.

Proof of equivalence:

(\Rightarrow) If $x \leq f(x)$, then $x \vee f(x) = f(x)$ by D1. Take $\alpha = f(x)$: then $f(x) = x \vee \alpha$. ✓

(\Leftarrow) If $f(x) = x \vee \alpha$ for some $\alpha \in L$, then $x \vee f(x) = x \vee (x \vee \alpha) = (x \vee x) \vee \alpha = x \vee \alpha = f(x)$ by A1, A3. By D1, $x \leq f(x)$. ✓

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence

A state sequence is a function $S: \mathbb{N} \rightarrow L$, written (S_0, S_1, S_2, \dots) , such that:

$S_{n+1} = S_n \vee \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Remark R2 — Terminology: State Sequence and Ascending Chain

In the order-theory literature, a sequence (S_n) satisfying $S_n \leq S_{n+1}$ for all n is called an ascending chain. The term "state sequence" is used here in place of "ascending chain" to align with the state-transition framing of ZP-D and ZP-E, where the same structure is introduced as sequences of system states. The two terms denote the same mathematical object. Readers familiar with order theory should read "state sequence" as "ascending chain". For concrete illustrations, see the ZP-A Illustrated Companion.

Theorem T3 — State Sequences are Monotone

For any state sequence (S_n) satisfying D3: $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$

Proof: By D3, $S_{n+1} = S_n \vee \alpha_n$. By D1, $S_n \leq S_n \vee \alpha_n \iff S_n \vee (S_n \vee \alpha_n) = S_n \vee \alpha_n$. By A1, $(S_n \vee S_n) \vee \alpha_n = S_n \vee \alpha_n$. By A3, $S_n \vee S_n = S_n$. Therefore $S_n \vee \alpha_n = S_{n+1}$. ✓

Monotonicity is a theorem, not a postulate. It is derived from A1–A3 via D3.

4.2 The Initial State

Every state sequence begins somewhere. T2 establishes $\perp \leq S_0$ for any initialisation — the bottom element is always below the starting point, whatever that starting point is. But T2 does not fix where S_0 sits; a sequence could legitimately begin above \perp . CC-1 closes this gap: we commit to initialising at the minimum. This is a modelling choice at ZP-A scope. In the AFAStructure lattices of ZP-J, T-EXEC derives it as a structural consequence rather than an assumption.

CC-1 — $S_0 = \perp$ | Derived in AFAStructure lattices (ZP-J T-EXEC); conditional at ZP-A scope

We commit to initialising every state sequence at the minimum of L : $S_0 = \perp$. This is not derived from A1–A4 — it is a modelling choice.

CC-1 — $S_0 = \perp$ | Derived in AFAStructure lattices (ZP-J T-EXEC); conditional at ZP-A scope

Under CC-1 and T3: $S_0 = \perp \leq S_1 \leq S_2 \leq \dots$

Note: By T2, $\perp \leq S_0$ for any initialisation — this holds unconditionally from A4. CC-1 strengthens this to equality: $S_0 = \perp$. The commitment is not needed to establish $\perp \leq S_0$; it is needed to fix the starting point precisely.

Status: DERIVED (given AFAStructure grounding) — ZP-J T-EXEC establishes $\text{IsQuineAtom}(\perp)$ as the unique structural identity; $S_0 = \perp$ follows as a structural consequence in any AFAStructure lattice. Modelling commitment at the ZP-A level (no AFAStructure assumption is made within this document).

V. The Self-Containment of \perp

5.1 Foundational Characterisation

The axioms A1–A4 establish \perp as the additive identity and algebraic minimum of L . The following conditional claim characterises its set-theoretic nature. R3 provides a structural route to DA-1 in ZP-E: CC-2 establishes that \perp has no external interpreter position, which — conditional on D7’s exhaustive static/executing dichotomy (ZP-E) as background — eliminates the static-description state for \perp . See R3 for the full dependency note.

Foundation note: The framework is stated over $ZF + AFA$ (Zermelo–Fraenkel set theory with Aczel’s Anti-Foundation Axiom). The classical Axiom of Foundation is replaced by AFA, which permits self-containing sets. This replacement is not an arbitrary modelling choice: $ZF + \text{Foundation}$ is incompatible with CC-2 (a well-founded \perp would admit an external interpreter, contradicting R3) and with ZP-C L-INF (bounded \in -rank contradicts unbounded surprisal of \perp). See ZP-E Remark R-AFA for the full cross-framework argument. The Axiom of Choice is not assumed.

Conditional Claim CC-2 — Self-Containment of \perp

The null state \perp is its own extension: the collection of all objects bearing the structural property of \perp is \perp itself.

Formally: $\perp = \{\perp\}$

Under $ZF + AFA$, \perp is a Quine atom — a set satisfying $x = \{x\}$. By set extensionality, any infinite collection of objects all indistinguishable under the structural property of \perp collapses to \perp itself. There is no multiplicity, only \perp .

This is a modeling commitment. It is not derived from A1–A4. It requires replacing the classical Axiom of Foundation with AFA in the metatheory.

Status: CONDITIONAL CLAIM — modeling commitment over $ZF + AFA$; not derived from A1–A4.

Cross-framework note: The replacement of Foundation by AFA is not an arbitrary choice — $ZF + \text{Foundation}$ is ruled out by R3 (a well-founded \perp would admit an external interpreter, contradicting CC-2) and by ZP-C L-INF (bounded \in -rank contradicts unbounded surprisal of \perp). AFA is the forced metatheoretic replacement; the specific form $\perp = \{\perp\}$ is the minimal Quine atom consistent with A4. See ZP-E Remark R-AFA for the full cross-framework argument.

Conditional Claim CC-2 — Self-Containment of \perp

Lean 4 scope: ZPA.lean verifies the algebraic structure A1–A4 and all derived results (T1–T3, CC-1). CC-2 is a metatheoretic commitment at the set-theoretic level. Lean’s bot field is a term of an abstract typeclass — a structural proxy for the algebraic role of \perp . Lean’s type theory (CIC) is well-founded by construction; Quine atoms cannot be realized as Lean terms. The set-theoretic content of CC-2 is stated as a prose-level commitment in ZF + AFA and is outside the scope of the Lean verification.

Remark R3 — CC-2 Eliminates the Static-Description State for \perp

A self-containing object has no external interpreter by structure: $\perp = \{\perp\}$ is its own interpretation. A description requires a describer distinct from the thing described; CC-2 admits no such distinction for \perp . Under the Turing model framework (D7, ZP-E), which partitions machine configurations into static-description states and executing states, the absence of any external interpreter position means \perp cannot occupy D7’s static-description category.

Dependency note: The inference from “no external interpreter” to “necessarily executing” uses D7’s exhaustiveness in the final step — that static-description and executing are the only two categories. D7 (ZP-E) supplies this exhaustiveness as the shared background framework. R3 provides the structural argument for why \perp engages D7’s transition; it does not independently derive DA-1 without D7. All three DA-1 paths in ZP-E share D7 as background. Their independence is among their arguments — CC-2/R3 (structural), L-INF (informational), K-incompressibility (AIT) — not from D7 itself.

VI. OQ-A1 — Sufficiency of Monotonicity

OQ-A1 — Sufficiency of Monotonicity [CLOSED within ZP — ZP-E T5]

Is the monotonicity constraint (T3) sufficient to characterise all valid state sequences, or are additional axioms required?

OQ-A1a: Is there algebraic reason to restrict α_n to join-irreducible elements (not expressible as joins of strictly smaller elements)?

OQ-A1b: Does the open-ended semilattice (without top element \top) permit unbounded ascending chains?

Status: CLOSED within ZP — Both sub-questions resolved by ZP-E Theorem T5 (Iterative Forcing Theorem) via AX-B1 from ZP-B. OQ-A1a: $\alpha_n = \varepsilon(S_n)$, the minimum viable deviation (ZP-specific). OQ-A1b: Within ZP, AX-B1’s binary constraint bounds ascending chains via T5. Note: in general semilattice theory without AX-B1, unbounded ascending chains are permitted — a bare join-semilattice imposes no bound. The closure here is a consequence of ZP’s binary existence commitment, not of A1–A4 alone.

Note on minimal element above \perp : Bounded chains do not in themselves guarantee a minimal element above \perp . In a dense structure, for any $\varepsilon > 0$, $\varepsilon/2$ also exists — chains may be bounded yet have no first step. The closure of OQ-A1b establishes that ascending chains are bounded within ZP; it does not establish that ε_0 exists as a specific minimal non-null element. That existence is a metric result, established by ZP-B’s 2-adic structure, which places 0 at infinite valuation and every non-zero element at finite valuation. The gap between them is not bridgeable by halving. This is outside the scope of A1–A4.

VII. Boundary Conditions

Export	Status / Receiving Document
(L, \vee, \perp) as join-semilattice	Derived (A1–A4) — ZP-D: algebraic structure of state space
\leq partial order (D1, T1)	Derived — ZP-D: ordering on states
Monotonicity of state sequences (T3)	Derived from A1–A3 — ZP-D: state layer ordering
\perp as global minimum (T2, CC-1)	Derived / Conditional — ZP-E: ontological grounding claim. CC-1 now derived in AFAStructure lattices via ZP-J T-EXEC.
$\perp = \{\perp\}$ self-containment (CC-2, R3)	Conditional / Remark — ZP-E: structural route to eliminating static-description state for \perp , given D7 exhaustiveness as background
No subtraction / additive ontology (R1)	Structural — ZP-C: no operation may reduce informational content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5

VIII. Validation Status

Component	Status / Notes
A1–A4 join-semilattice axioms	Valid — Axioms; self-contained
\leq partial order (D1, T1)	Valid — Derived from A1–A3
\perp as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1–A3 and D3
CC-1: $S_0 = \perp$	DERIVED (given AFAStructure grounding, ZP-J T-EXEC) — structural consequence in any AFAStructure lattice. Modelling commitment at ZP-A level without AFAStructure assumption.
CC-2: $\perp = \{\perp\}$	Conditional Claim — modeling commitment over ZF + AFA; not derived from A1–A4
ZF + AFA foundation (no AC)	Meta-theoretic — framework-wide; required for CC-2
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5