

THE ZERO PARADOX

ZP-A: Lattice Algebra

Version 1.3 | April 2026

Supersedes v1.2 | D2 quantifier clarified; T3 symbol corrected; CC-1 reframed

This document is self-contained within abstract algebra. No topology, probability, or Hilbert space is imported. Every claim is provable using only the tools of semilattice theory. Cross-framework connections are deferred to ZP-E.

Version 1.3 changes: (1) Definition D2: the equivalence statement now makes explicit that α depends on x — "for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$ ". (2) Theorem T3 proof: replaced the single spelled-out "iff" with \iff for consistency. (3) CC-1: removed circular conditional framing; reframed as a direct modelling commitment; corrected the consequence chain to $S_0 = \perp \leq S_1 \leq \dots$; replaced informal "constituent" with direct T2 reference.

Version 1.2 changes: (1) Definition D1: the notation $:\iff$ (non-standard) replaced by the standard definitional framing "define the relation \leq by". (2) Definition D2: the equivalence between $x \leq f(x)$ and $f(x) = x \vee \alpha$ is now accompanied by an explicit two-line proof of both directions.

Version 1.1 change: Theorem T4 reclassified as Conditional Claim CC-1. The v1.0 label "Theorem" was imprecise: the result holds only given the assumption that the state sequence is initialised at the minimum of L . This assumption is not derived from A1–A4 — it is a modelling commitment.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple: (L, \vee, \perp)

L is a non-empty set (the carrier set of states). $\vee : L \times L \rightarrow L$ is a binary operation called join. $\perp \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom

A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$

A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$

A3 — Idempotency: $x \vee x = x$ for all $x \in L$

A4 — Identity (Additive): $\perp \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes \perp the additive identity of the algebra: the element that contributes nothing to a join and is therefore present in every state as the neutral

constituent.

II. The Induced Partial Order

2.1 Definition of \leq

Definition D1 — Lattice Order

For $x, y \in L$, define the relation \leq by:

$$x \leq y \iff x \vee y = y$$

Theorem T1 — \leq is a Partial Order

Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓

Antisymmetry: if $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓

Transitivity: if $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 \perp is the Least Element

Theorem T2 — \perp is a Global Minimum under \leq

For all $x \in L$: $\perp \leq x$

Proof: By A4, $\perp \vee x = x$. By D1, this is the definition of $\perp \leq x$. ✓

T2 is the algebraic statement of the foundational claim: \perp is not a void that states depart from — it is the minimum element that every state sits above. Since $\perp \leq x$ for all x , and join accumulates from the bottom, \perp is algebraically present in every element of L .

III. The Additive Ontology

3.1 No Subtraction Operator

Remark R1 — Join-Semilattice vs. Lattice

A full lattice $(L, \vee, \wedge, \perp, \top)$ includes a meet operator \wedge and a top element \top . The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — the removal of informational content from a state. The additive ontology requires that no operation decreases informational content.

3.2 Join is the Only State Transition

Definition D2 — State Transition

A state transition is any function $f: L \rightarrow L$ such that $x \leq f(x)$ for all $x \in L$.

Equivalently, for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$.

Definition D2 — State Transition

Proof of equivalence:

(\Rightarrow) If $x \leq f(x)$, then $x \vee f(x) = f(x)$ by D1. Take $\alpha = f(x)$: then $f(x) = x \vee \alpha$. \checkmark

(\Leftarrow) If $f(x) = x \vee \alpha$ for some $\alpha \in L$, then $x \vee f(x) = x \vee (x \vee \alpha) = (x \vee x) \vee \alpha = x \vee \alpha = f(x)$ by A1, A3. By D1, $x \leq f(x)$. \checkmark

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence

A state sequence is a function $S: \mathbb{N} \rightarrow L$, written (S_0, S_1, S_2, \dots) , such that:

$S_{n+1} = S_n \vee \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Theorem T3 — State Sequences are Monotone

For any state sequence (S_n) satisfying D3: $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$

Proof: By D3, $S_{n+1} = S_n \vee \alpha_n$. By D1, $S_n \leq S_n \vee \alpha_n \Leftrightarrow S_n \vee (S_n \vee \alpha_n) = S_n \vee \alpha_n$. By A1, $(S_n \vee S_n) \vee \alpha_n = S_n \vee \alpha_n$. By A3, $S_n \vee S_n = S_n$. Therefore $S_n \vee \alpha_n = S_{n+1}$. \checkmark

Monotonicity is a theorem, not a postulate. It is derived from A1-A3 via D3.

4.2 The Initial State

Conditional Claim CC-1 — $S_0 = \perp$ (Reclassified from T4 in v1.0)

We commit to initialising every state sequence at the minimum of L : $S_0 = \perp$. This is not derived from A1-A4 — it is a modelling choice.

Under CC-1 and T3: $S_0 = \perp \leq S_1 \leq S_2 \leq \dots$

Corollary: By T2, $\perp \leq S_n$ for all n . Every state in the sequence sits above the global minimum.

Status: CONDITIONAL CLAIM — modelling commitment; not derived from A1-A4.

V. Open Question OQ-A1

Open Question OQ-A1 — Sufficiency of Monotonicity

Is the monotonicity constraint (T3) sufficient to characterise all valid state sequences, or are additional axioms required?

Open Question OQ-A1 — Sufficiency of Monotonicity

OQ-A1a: Is there algebraic reason to restrict α_n to join-irreducible elements (not expressible as joins of strictly smaller elements)?

OQ-A1b: Does the open-ended semilattice (without top element \top) permit unbounded ascending chains?

Status: Open within ZP-A. Closed by ZP-E Theorem T5 (Iterative Forcing Theorem) via AX-B1 from ZP-B.

VI. Boundary Conditions

Export	Status / Receiving Document
(L, v, \perp) as join-semilattice	Derived (A1-A4) — ZP-D: algebraic structure of state space
\leq partial order (D1, T1)	Derived — ZP-D: ordering on states
Monotonicity of state sequences (T3)	Derived from A1-A3 — ZP-D: state layer ordering
\perp as global minimum (T2, CC-1)	Derived / Conditional — ZP-E: ontological grounding claim
No subtraction / additive ontology (R1)	Structural — ZP-C: no operation may reduce informational content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5

VII. Validation Status

Component	Status / Notes
A1-A4 join-semilattice axioms	Valid — Axioms; self-contained
\leq partial order (D1, T1)	Valid — Derived from A1-A3
\perp as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1-A3 and D3
CC-1: $S_0 = \perp$	Conditional Claim — modelling commitment; not derived from A1-A4
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5