THE ZERO PARADOX

ZP-A: Lattice Algebra

Version 1.4 | April 2026

Supersedes v1.3 | OQ-A1 status corrected to closed

This document is self-contained within abstract algebra. No topology, probability, or Hilbert space is imported. Every claim is provable using only the tools of semilattice theory. Cross-framework connections are deferred to ZP-E.

Version 1.4 change: OQ-A1 section heading and box label corrected from "Open Question" to "CLOSED". The resolution was already recorded in the status line (closed by ZP-E T5 via AX-B1) but the section header was misleading. Status line expanded to answer both sub-questions explicitly.

Version 1.3 changes: (1) Definition D2: the equivalence statement now makes explicit that α depends on x — "for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$". (2) Theorem T3 proof: replaced the single spelled-out "iff" with $\Longleftrightarrow$ for consistency. (3) CC-1: removed circular conditional framing; reframed as a direct modelling commitment; corrected the consequence chain to $S_0 = \bot \leq S_1 \leq \ldots$; replaced informal "constituent" with direct T2 reference.

Version 1.2 changes: (1) Definition D1: the notation $:\Longleftrightarrow$ (non-standard) replaced by the standard definitional framing "define the relation $\leq$ by". (2) Definition D2: the equivalence between $x \leq f(x)$ and $f(x) = x \vee \alpha$ is now accompanied by an explicit two-line proof of both directions.

Version 1.1 change: Theorem T4 reclassified as Conditional Claim CC-1. The v1.0 label "Theorem" was imprecise: the result holds only given the assumption that the state sequence is initialised at the minimum of L. This assumption is not derived from A1–A4 — it is a modelling commitment.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple: $(L, \vee, \bot)$

L is a non-empty set (the carrier set of states). $\vee : L \times L \to L$ is a binary operation called join. $\bot \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom
A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$
A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$

Axiom Block A — Join-Semilattice with Bottom
A3 — Idempotency: $x \vee x = x$ for all $x \in L$
A4 — Identity (Additive): $\bot \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes ⊥ the additive identity of the algebra: the element that contributes nothing to a join and is therefore present in every state as the neutral constituent.

II. The Induced Partial Order

2.1 Definition of ≤

Definition D1 — Lattice Order
For $x, y \in L$, define the relation $\leq$ by:
$x \leq y \Longleftrightarrow x \vee y = y$

Theorem T1 — ≤ is a Partial Order
Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓
Antisymmetry: if $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓
Transitivity: if $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 ⊥ is the Least Element

Theorem T2 — ⊥ is a Global Minimum under ≤
For all $x \in L$: $\bot \leq x$
Proof: By A4, $\bot \vee x = x$. By D1, this is the definition of $\bot \leq x$. ✓

T2 is the algebraic statement of the foundational claim: ⊥ is not a void that states depart from — it is the minimum element that every state sits above. Since $\bot \leq x$ for all x, and join accumulates from the bottom, ⊥ is algebraically present in every element of L.

III. The Additive Ontology

3.1 No Subtraction Operator

Remark R1 — Join-Semilattice vs. Lattice
A full lattice $(L, \vee, \wedge, \bot, \top)$ includes a meet operator $\wedge$ and a top element $\top$. The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — the removal of informational content from a state. The additive ontology requires that no operation decreases informational content.

3.2 Join is the Only State Transition

Definition D2 — State Transition
A state transition is any function f: L → L such that $x \le f(x)$ for all $x \in L$.
Equivalently, for each $x \in L$, $f(x) = x \lor \alpha$ for some $\alpha \in L$.
Proof of equivalence:
(⇒) If $x \le f(x)$, then $x \lor f(x) = f(x)$ by D1. Take $\alpha = f(x)$: then $f(x) = x \lor \alpha$. ✓
(⇐) If $f(x) = x \lor \alpha$ for some $\alpha \in L$, then $x \lor f(x) = x \lor (x \lor \alpha) = (x \lor x) \lor \alpha = x \lor \alpha = f(x)$ by A1, A3. By D1, $x \le f(x)$. ✓

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence
A state sequence is a function S: ℕ → L, written $(S_0, S_1, S_2, \ldots)$, such that:
$S_{n+1} = S_n \lor \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Theorem T3 — State Sequences are Monotone
For any state sequence (S_n) satisfying D3: $S_n \le S_{n+1}$ for all $n \in \mathbb{N}$
Proof: By D3, $S_{n+1} = S_n \lor \alpha_n$. By D1, $S_n \le S_n \lor \alpha_n \iff S_n \lor (S_n \lor \alpha_n) = S_n \lor \alpha_n$. By A1, $(S_n \lor S_n) \lor \alpha_n = S_n \lor \alpha_n$. By A3, $S_n \lor S_n = S_n$. Therefore $S_n \lor \alpha_n = S_{n+1}$. ✓
Monotonicity is a theorem, not a postulate. It is derived from A1–A3 via D3.

4.2 The Initial State

Conditional Claim CC-1 — $S_0 = \bot$ (Reclassified from T4 in v1.0)
We commit to initialising every state sequence at the minimum of L: $S_0 = \bot$. This is not derived from A1–A4 — it is a modelling choice.
Under CC-1 and T3: $S_0 = \bot \le S_1 \le S_2 \le \ldots$
Corollary: By T2, $\bot \le S_n$ for all n. Every state in the sequence sits above the global minimum.
Status: CONDITIONAL CLAIM — modelling commitment; not derived from A1–A4.

V. OQ-A1 — Sufficiency of Monotonicity

OQ-A1 — Sufficiency of Monotonicity [CLOSED — ZP-E T5]
Is the monotonicity constraint (T3) sufficient to characterise all valid state sequences, or are additional axioms required?
OQ-A1a: Is there algebraic reason to restrict α_n to join-irreducible elements (not expressible as joins of strictly smaller elements)?
OQ-A1b: Does the open-ended semilattice (without top element ⊤) permit unbounded ascending chains?
Status: CLOSED — Both sub-questions resolved by ZP-E Theorem T5 (Iterative Forcing Theorem) via AX-B1 from ZP-B. OQ-A1a: $\alpha_n = \varepsilon(S_n)$, the minimum viable deviation. OQ-A1b: AX-B1's binary constraint bounds ascending chains.

VI. Boundary Conditions

Export	Status / Receiving Document
(L, ∨, ⊥) as join-semilattice	Derived (A1–A4) — ZP-D: algebraic structure of state space
≤ partial order (D1, T1)	Derived — ZP-D: ordering on states
Monotonicity of state sequences (T3)	Derived from A1–A3 — ZP-D: state layer ordering
⊥ as global minimum (T2, CC-1)	Derived / Conditional — ZP-E: ontological grounding claim
No subtraction / additive ontology (R1)	Structural — ZP-C: no operation may reduce informational content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5

VII. Validation Status

Component	Status / Notes
A1–A4 join-semilattice axioms	Valid — Axioms; self-contained
≤ partial order (D1, T1)	Valid — Derived from A1–A3
⊥ as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1–A3 and D3
CC-1: $S_0 = \bot$	Conditional Claim — modelling commitment; not derived from A1–A4

Component	Status / Notes
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5