THE ZERO PARADOX

ZP-A: Lattice Algebra

Version 1.6 | April 2026

Supersedes v1.5 | CC-2 (Self-Containment of ⊥) and R3 added; foundation note (ZF + AFA; AC not assumed) added

This document is self-contained within abstract algebra. No topology, probability, or Hilbert space is imported. Every claim is provable using only the tools of semilattice theory. Cross-framework connections are deferred to ZP-E.

Illustrated Companion: A paired ZP-A Illustrated Companion document provides concrete examples and visual intuitions for the results in this document. Examples are kept separate from the formal layers to distinguish illustrative material from proofs. The companion is a reading aid; no proof-critical judgements should be drawn from examples alone.

Version 1.6 changes: CC-2 (Self-Containment of ⊥) added as a new modeling commitment: $\bot = \{\bot\}$ is a Quine atom under ZF + AFA (Aczel's Anti-Foundation Axiom). R3 added immediately following CC-2, establishing that DA-1 in ZP-E follows from CC-2 as a derivation rather than a design principle. Foundation note added: the framework is stated over ZF + AFA; the Axiom of Choice is not assumed. Section V dedicated to self-containment of ⊥; OQ-A1 renumbered VI, Boundary Conditions VII, Validation Status VIII.

Version 1.5 changes: (1) Theorem/Proposition/Lemma hierarchy applied throughout: T1 relabelled Proposition (partial order properties are infrastructure), T2 relabelled Lemma (the global minimum result is a stepping stone for CC-1 and T3). T3 retains Theorem (monotonicity is the primary result of ZP-A). (2) Remark R2 added after D3 connecting the term "state sequence" to the standard order-theory term "ascending chain". (3) CC-1 corollary reworded to make explicit that T2 gives $\bot \leq S_0$ for any initialisation; CC-1 strengthens this to equality.

Version 1.4 change: OQ-A1 section heading and box label corrected from "Open Question" to "CLOSED". The resolution was already recorded in the status line (closed by ZP-E T5 via AX-B1) but the section header was misleading. Status line expanded to answer both sub-questions explicitly.

Version 1.3 changes: (1) Definition D2: the equivalence statement now makes explicit that α depends on x — "for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$". (2) Theorem T3 proof: replaced the single spelled-out "iff" with $\Longleftrightarrow$ for consistency. (3) CC-1: removed circular conditional framing; reframed as a direct modelling commitment; corrected the consequence chain to $S_0 = \bot \leq S_1 \leq \ldots$; replaced informal "constituent" with direct T2 reference.

Version 1.2 changes: (1) Definition D1: the notation $:\Longleftrightarrow$ (non-standard) replaced by the standard definitional framing "define the relation ≤ by". (2) Definition D2: the equivalence between $x \leq f(x)$ and $f(x) = x \vee \alpha$ is now accompanied by an explicit two-line proof of both directions.

Version 1.1 change: Theorem T4 reclassified as Conditional Claim CC-1. The v1.0 label "Theorem" was imprecise: the result holds only given the assumption that the state sequence is initialised at the minimum of L. This assumption is not derived from A1–A4 — it is a modelling commitment.

I. Primitives and Axioms

1.1 Signature

The algebraic signature of the Zero Paradox state space is a triple: $(L, \vee, \bot)$

L is a non-empty set (the carrier set of states). $\vee : L \times L \to L$ is a binary operation called join. $\bot \in L$ is a distinguished constant called the bottom element.

Axiom Block A — Join-Semilattice with Bottom
A1 — Associativity: $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in L$
A2 — Commutativity: $x \vee y = y \vee x$ for all $x, y \in L$
A3 — Idempotency: $x \vee x = x$ for all $x \in L$
A4 — Identity (Additive): $\bot \vee x = x$ for all $x \in L$

A4 is the load-bearing axiom. It makes $\bot$ the additive identity of the algebra: the element that contributes nothing to a join and is therefore present in every state as the neutral constituent.

II. The Induced Partial Order

2.1 Definition of ≤

Definition D1 — Lattice Order
For $x, y \in L$, define the relation $\leq$ by:
$x \leq y \iff x \vee y = y$

Proposition T1 — ≤ is a Partial Order
Reflexivity: $x \leq x$ — by A3, $x \vee x = x$. ✓
Antisymmetry: if $x \leq y$ and $y \leq x$, then $x \vee y = y$ and $y \vee x = x$. By A2, $y = x \vee y = y \vee x = x$. ✓
Transitivity: if $x \leq y$ and $y \leq z$, then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$, so $x \leq z$. ✓

2.2 ⊥ is the Least Element

Lemma T2 — ⊥ is a Global Minimum under ≤
For all $x \in L$: $\bot \leq x$
Proof: By A4, $\bot \vee x = x$. By D1, this is the definition of $\bot \leq x$. ✓

T2 is the algebraic statement of the foundational claim: $\bot$ is not a void that states depart from — it is the minimum element that every state sits above. Since $\bot \leq x$ for all x, and join accumulates from the bottom, $\bot$ is algebraically present in every element of L.

III. The Additive Ontology

3.1 No Subtraction Operator

Remark R1 — Join-Semilattice vs. Lattice

A full lattice $(L, \vee, \wedge, \bot, \top)$ includes a meet operator $\wedge$ and a top element $\top$. The Zero Paradox restricts to the join-semilattice with bottom. The meet operator is excluded because it would allow state reduction — the removal of informational content from a state. The additive ontology requires that no operation decreases informational content.

3.2 Join is the Only State Transition

Definition D2 — State Transition

A state transition is any function $f: L \to L$ such that $x \leq f(x)$ for all $x \in L$.

Equivalently, for each $x \in L$, $f(x) = x \vee \alpha$ for some $\alpha \in L$.

Proof of equivalence:

($\Rightarrow$) If $x \leq f(x)$, then $x \vee f(x) = f(x)$ by D1. Take $\alpha = f(x)$: then $f(x) = x \vee \alpha$. ✓

($\Leftarrow$) If $f(x) = x \vee \alpha$ for some $\alpha \in L$, then $x \vee f(x) = x \vee (x \vee \alpha) = (x \vee x) \vee \alpha = x \vee \alpha = f(x)$ by A1, A3. By D1, $x \leq f(x)$. ✓

IV. Monotonicity of State Sequences

4.1 State Sequences

Definition D3 — State Sequence

A state sequence is a function $S: \mathbb{N} \to L$, written $(S_0, S_1, S_2, \ldots)$, such that:

$S_{n+1} = S_n \vee \alpha_n$ for some $\alpha_n \in L$, for all $n \in \mathbb{N}$

Remark R2 — Terminology: State Sequence and Ascending Chain

In the order-theory literature, a sequence (S_n) satisfying $S_n \leq S_{n+1}$ for all n is called an ascending chain. The term "state sequence" is used here in place of "ascending chain" to align with the state-transition framing of ZP-D and ZP-E, where the same structure is introduced as sequences of system states. The two terms denote the same mathematical object. Readers familiar with order theory should read "state sequence" as "ascending chain". For concrete illustrations, see the ZP-A Illustrated Companion.

Theorem T3 — State Sequences are Monotone

For any state sequence (S_n) satisfying D3: $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$

Proof: By D3, $S_{n+1} = S_n \vee \alpha_n$. By D1, $S_n \leq S_n \vee \alpha_n \iff S_n \vee (S_n \vee \alpha_n) = S_n \vee \alpha_n$. By A1, $(S_n \vee S_n) \vee \alpha_n = S_n \vee \alpha_n$. By A3, $S_n \vee S_n = S_n$. Therefore $S_n \vee \alpha_n = S_{n+1}$. ✓

Monotonicity is a theorem, not a postulate. It is derived from A1–A3 via D3.

4.2 The Initial State

Conditional Claim CC-1 — $S_0 = \bot$ (Reclassified from T4 in v1.0)
We commit to initialising every state sequence at the minimum of L: $S_0 = \bot$. This is not derived from A1–A4 — it is a modelling choice.
Under CC-1 and T3: $S_0 = \bot \leq S_1 \leq S_2 \leq \ldots$
Note: By T2, $\bot \leq S_0$ for any initialisation — this holds unconditionally from A4. CC-1 strengthens this to equality: $S_0 = \bot$. The commitment is not needed to establish $\bot \leq S_0$; it is needed to fix the starting point precisely.
Status: CONDITIONAL CLAIM — modelling commitment; not derived from A1–A4.

V. The Self-Containment of ⊥

5.1 Foundational Characterisation

The axioms A1–A4 establish ⊥ as the additive identity and algebraic minimum of L. The following conditional claim characterises its set-theoretic nature and grounds DA-1 in ZP-E as a derivation rather than a design principle.

Foundation note: The framework is stated over ZF + AFA (Zermelo–Fraenkel set theory with Aczel's Anti-Foundation Axiom). The classical Axiom of Foundation is replaced by AFA, which permits self-containing sets. The Axiom of Choice is not assumed: T-SNAP is the unique forced first differentiation, not a selection over indistinguishable ⊥ instances.

Conditional Claim CC-2 — Self-Containment of ⊥
The null state ⊥ is its own extension: the collection of all objects bearing the structural property of ⊥ is ⊥ itself.
Formally: $\bot = \{\bot\}$
Under ZF + AFA, ⊥ is a Quine atom — a set satisfying $x = \{x\}$. By set extensionality, any infinite collection of objects all indistinguishable under the structural property of ⊥ collapses to ⊥ itself. There is no multiplicity, only ⊥.
This is a modeling commitment. It is not derived from A1–A4. It requires replacing the classical Axiom of Foundation with AFA in the metatheory.
Status: CONDITIONAL CLAIM — modeling commitment over ZF + AFA; not derived from A1–A4.
Lean 4 scope: ZPA.lean verifies the algebraic structure A1–A4 and all derived results (T1–T3, CC-1). CC-2 is a metatheoretic commitment at the set-theoretic level. Lean's bot field is a term of an abstract typeclass — a structural proxy for the algebraic role of ⊥. Lean's type theory (CIC) is well-founded by construction; Quine atoms cannot be realized as Lean terms. The set-theoretic content of CC-2 is stated as a prose-level commitment in ZF + AFA and is outside the scope of the Lean verification.

Remark R3 — DA-1 Follows from CC-2
A self-containing object has no external interpreter by structure: $\bot = \{\bot\}$ is its own interpretation. A description requires a describer distinct from the thing described; CC-2 admits no such distinction for $\bot$.
Therefore $\bot$ at P_0 cannot be a static description awaiting external instantiation. This is the formal basis for DA-1 in ZP-E: the claim that instantiation at P_0 is a live execution event becomes a derivation from CC-2, not a freestanding design principle.

VI. OQ-A1 — Sufficiency of Monotonicity

OQ-A1 — Sufficiency of Monotonicity [CLOSED — ZP-E T5]
Is the monotonicity constraint (T3) sufficient to characterise all valid state sequences, or are additional axioms required?
OQ-A1a: Is there algebraic reason to restrict α_n to join-irreducible elements (not expressible as joins of strictly smaller elements)?
OQ-A1b: Does the open-ended semilattice (without top element $\top$) permit unbounded ascending chains?
Status: CLOSED — Both sub-questions resolved by ZP-E Theorem T5 (Iterative Forcing Theorem) via AX-B1 from ZP-B. OQ-A1a: $\alpha_n = \varepsilon(S_n)$, the minimum viable deviation. OQ-A1b: AX-B1's binary constraint bounds ascending chains.

VII. Boundary Conditions

Export	Status / Receiving Document
$(L, \vee, \bot)$ as join-semilattice	Derived (A1–A4) — ZP-D: algebraic structure of state space
$\leq$ partial order (D1, T1)	Derived — ZP-D: ordering on states
Monotonicity of state sequences (T3)	Derived from A1–A3 — ZP-D: state layer ordering
$\bot$ as global minimum (T2, CC-1)	Derived / Conditional — ZP-E: ontological grounding claim
$\bot = \{\bot\}$ self-containment (CC-2, R3)	Conditional / Remark — ZP-E: basis for DA-1 derivation
No subtraction / additive ontology (R1)	Structural — ZP-C: no operation may reduce informational content
OQ-A1 — increment selection	Open within ZP-A; closed by ZP-E T5

VIII. Validation Status

Component	Status / Notes
A1–A4 join-semilattice axioms	Valid — Axioms; self-contained
$\leq$ partial order (D1, T1)	Valid — Derived from A1–A3

Component	Status / Notes
$\bot$ as least element (T2)	Valid — Derived from A4 and D1
Additive ontology / no subtraction (R1)	Valid — Structural; signature restriction
State transition as join (D2)	Valid — Defined; consistent with signature
Monotonicity of state sequences (T3)	Valid — Derived from A1–A3 and D3
CC-1: $S_0 = \bot$	Conditional Claim — modelling commitment; not derived from A1–A4
CC-2: $\bot = \{\bot\}$	Conditional Claim — modeling commitment over ZF + AFA; not derived from A1–A4
ZF + AFA foundation (no AC)	Meta-theoretic — framework-wide; required for CC-2
OQ-A1: Sufficiency of monotonicity	Open within ZP-A; closed by ZP-E T5