THE ZERO PARADOX

ZP-B: p-Adic Topology

Version 1.3 | April 2026

Supersedes v1.2 | Theorem/Proposition hierarchy applied: T1, T2, T5 relabelled Proposition

This document is self-contained within p-adic analysis and topology. No abstract algebra from ZP-A, no probability, and no Hilbert space is imported. Cross-framework connections are deferred to ZP-D and ZP-E.

Illustrated Companion: A paired ZP-B Illustrated Companion provides concrete examples and visual intuitions for the results here. Examples are kept separate from the formal layers to distinguish illustrative material from proofs.

Version 1.3 change: Theorem/Proposition hierarchy applied. T1 (Strong Triangle Inequality) and T2 (Every Ball is Clopen) relabelled Proposition — both are well-known infrastructure results of p-adic analysis, not primary claims of this framework. T5 (Total Disconnectedness) relabelled Proposition — it is load-bearing infrastructure for C3. T0 ($p=2$ is uniquely derived) and T3 (Topological Isolation of 0) retain Theorem labels as the primary claims of ZP-B.

Version 1.2 changes: T0 strengthened with MP-1; C2 fixed to derive from T2 only; T4 reclassified as C3 (corollary of T5).

I. The Foundational Distinction

1.1 The Binary Existence Axiom

Axiom AX-B1 — Binary Existence
The foundational distinction of the Zero Paradox framework is binary: a state either exists or it does not. There is no third option at this level.
0 — non-existence (the Null State, corresponding to ⊥ in ZP-A)
1 — existence (the First Atomic State, the minimal non-zero element)
Status: AXIOM. This is the only non-topological commitment in ZP-B. It precedes p-adic analysis and is the premise from which the field selection is derived.
Scope: AX-B1 asserts the structure of the ontological distinction, not its physical realisation. It is invariant across all instantiations.

1.2 The Minimality Principle

Principle MP-1 — Minimality of Representation
The representational base of the framework must be the minimum base capable of encoding the ontological distinction of AX-B1 without redundancy and without information loss.
Redundancy: a base $p >$ minimum introduces representational states with no ontological counterpart, violating parsimony.
Information loss: a base $p <$ minimum cannot distinguish all ontologically distinct states, violating faithfulness.
Status: PRINCIPLE — methodological commitment. Given AX-B1 (two ontological states) and MP-1 (minimum sufficient base), the representational base is uniquely determined as 2.

1.3 Derivation of p = 2

Theorem T0 — p = 2 is the Unique Minimum Sufficient Representational Base
Given AX-B1 and MP-1, the p-adic field appropriate for the Zero Paradox framework is Q_2.
Note on MP-1: MP-1 is a design commitment — the choice to use the minimum sufficient base. Any Q_p for prime $p \geq 2$ contains elements 0 and 1 capable of representing the AX-B1 distinction; the choice of minimum base is what MP-1 encodes. T0 is a valid derivation from AX-B1 and MP-1, but MP-1 is the load-bearing design choice, not a mathematical necessity. OQ-B1 is closed given MP-1.
Proof:
Step 1 — AX-B1 establishes exactly two ontological states: non-existence (0) and existence (1).
Step 2 — A p-adic field Q_p uses coefficients from $\{0, 1, \ldots, p-1\}$. The minimum base p capable of representing exactly two distinct values without redundancy is $p = 2$, with coefficient set $\{0, 1\}$. One coefficient per ontological state; no unused coefficients.
Step 3 — $p = 1$ has only one coefficient value $\{0\}$. Cannot distinguish existence from non-existence. Fails faithfulness (MP-1).
Step 4 — $p > 2$: coefficient set $\{0, \ldots, p-1\}$ contains values with no ontological counterpart. Violates no-redundancy condition of MP-1.
Step 5 — $p = 2$ is the unique prime satisfying both conditions simultaneously.
Step 6 — The binary branching at every level of Q_2's ball structure reflects the eventual binary resolution of any representational complexity.
Therefore $p = 2$. Status: DERIVED given MP-1 (design commitment). OQ-B1 closed. ✓

II. The 2-Adic Field

2.1 The 2-Adic Absolute Value

Fix $p = 2$ (derived in T0). Every non-zero rational $q \in \mathbb{Q}$ can be written uniquely as $q = 2^v \cdot (a/b)$ where $v \in \mathbb{Z}$ and a, b are integers not divisible by 2. The integer v is the 2-adic valuation $v_2(q)$. By convention, $v_2(0) =$

$+\infty$.

Definition D1 — 2-Adic Absolute Value

For $q \in \mathbb{Q}$: $|q|_2 = 2^{-v_2(q)}$ for $q \neq 0$; $|0|_2 = 0$

Elements with high powers of 2 are considered small under $|\cdot|_2$. Elements with no factor of 2 have $|\cdot|_2 = 1$.

2.2 The 2-Adic Field Q_2

Q_2 is the completion of $\mathbb{Q}$ under the metric induced by $|\cdot|_2$. Elements of Q_2 are formal power series in 2: $x = \sum_{n=v}^{\infty} a_n \cdot 2^n$ where $a_n \in \{0,1\}$. The coefficients $a_n \in \{0,1\}$ are precisely the binary values of AX-B1.

2.3 The Minimum Viable Deviation ε_0

Definition D5 — Minimum Viable Deviation ε_0

$\varepsilon_0 = 2^k$ for some integer k, where k is the maximum valuation accessible in the instantiation.

Structural role (universal): ε_0 is always the first element crossed by the Snap. Fixed by the structure of Q_2 and AX-B1.

Numerical value (contingent): determined by physical constants of the instantiation. Planck-scale quantities in our universe.

Status: DEFINED — universe-contingent parameter.

III. The Ultrametric

Definition D2 — 2-Adic Metric

For $x, y \in Q_2$: $d(x, y) = |x - y|_2$

Proposition T1 — Strong Triangle Inequality (Ultrametric)

For all $x, y, z \in Q_2$: $d(x, z) \leq \max(d(x, y), d(y, z))$

Proof: Write $x - z = (x - y) + (y - z)$. The ultrametric property of v_2 gives $v_2(a+b) \geq \min(v_2(a), v_2(b))$, from which $|a+b|_2 \leq \max(|a|_2, |b|_2)$. Apply with $a = x-y$ and $b = y-z$. ✓

Corollary C1 — All Triangles are Isosceles

If $d(x,y) \neq d(y,z)$, then $d(x,z) = \max(d(x,y), d(y,z))$.

Proof: Suppose $d(x,y) < d(y,z)$. By T1, $d(x,z) \leq d(y,z)$. Also $d(y,z) \leq \max(d(y,x), d(x,z)) = d(x,z)$ since $d(x,y) < d(y,z)$. Therefore $d(x,z) = d(y,z)$. ✓

3.2 Clopen Ball Structure

Definition D3 — Ball in Q_2

$B(a, r) = \{ x \in Q_2 : d(x, a) \leq r \}$ (closed ball)

$B°(a, r) = \{ x \in Q_2 : d(x, a) < r \}$ (open ball)

Proposition T2 — Every Ball is Clopen

In Q_2, every closed ball is also open and every open ball is also closed.

Proof (closed ball is open): Let $y \in B(a, r)$. For any $z \in B(y, r)$, T1 gives $d(z, a) \leq \max(d(z,y), d(y,a)) \leq r$. So $B(y,r) \subseteq B(a,r)$. Every point is an interior point. ✓

Proof (open ball is closed): Let $(x_n) \to x$ with all $x_n \in B°(a,r)$. Ball radii in Q_2 are discrete (powers of 2), so $d(x,a) < r$ holds in the limit. Thus $x \in B°(a,r)$. ✓

Corollary C2 — Disjoint Balls Do Not Communicate [v1.2: derived from T2 only]

If $B(a, r)$ and $B(b, r)$ are disjoint ($d(a,b) > r$), then no continuous path exists from any point in $B(a, r)$ to any point in $B(b, r)$.

Proof: By T2, $B(a, r)$ is clopen in Q_2. Any continuous $f: [0,1] \to Q_2$ with $f(0) \in B(a,r)$ and $f(1) \in B(b,r)$ would require f to map the connected set $[0,1]$ onto a subset intersecting both $B(a,r)$ and its clopen complement. The preimage of a clopen set under a continuous function is clopen in $[0,1]$. Since $[0,1]$ is connected, the preimage is either empty or all of $[0,1]$. It cannot be all of $[0,1]$ (since $f(1) \notin B(a,r)$) and cannot be empty (since $f(0) \in B(a,r)$). Contradiction. ✓

IV. Topological Isolation of Zero

Theorem T3 — Topological Isolation of 0

For any $r = 2^{-k}$, the ball $B(0,r) = \{ x \in Q_2 : v_2(x) \geq k \}$. Any x outside this ball has $d(0,x) \geq 2^{-k+1} > r$. $B(0,r)$ and its complement are separated by a gap of at least 2^{-k}.

The transition from 0 to any non-zero element is a discrete jump across a clopen boundary — the topological identity of the Snap.

Relationship to ε_0: $\varepsilon_0 = 2^k$ is the smallest non-zero element outside the tightest ball around 0. The Snap crosses exactly this gap. ✓

V. Topological Structure of Q₂

5.1 Total Disconnectedness — proven before C3

Proposition T5 — Q_2 is Totally Disconnected

The only connected subsets of Q_2 are singletons.

Proposition T5 — Q_2 is Totally Disconnected

Proof: Let $S \subseteq Q_2$ contain two distinct points a, b with $d(a,b) = r > 0$. Choose s with $0 < s < r$. By T2, B(a,s) is clopen. Then $S = [S \cap B(a,s)] \cup [S \setminus B(a,s)]$ is a separation of S into two disjoint non-empty clopen sets. Therefore S is not connected. Since S was arbitrary, the only connected subsets are singletons. ✓

5.2 Topological Irreversibility of the Snap

Definition D4 — Topological Irreversibility

A transition from a to b in a topological space X is topologically irreversible if there exists no continuous path $\gamma: [0,1] \to X$ with $\gamma(0) = b$ and $\gamma(1) = a$.

Corollary C3 — The Snap is Topologically Irreversible [reclassified from T4 in v1.1]

Let $x \in Q_2$ with $x \neq 0$. There exists no continuous path $\gamma: [0,1] \to Q_2$ with $\gamma(0) = x$ and $\gamma(1) = 0$.

Proof: By T5, Q_2 is totally disconnected. A continuous path γ with $\gamma(0) = x \neq 0$ and $\gamma(1) = 0$ would require $\gamma([0,1])$ to be a connected subset of Q_2 containing two distinct points. By T5, no such connected subset exists. ✓

Derivation chain: T1 → T2 → T5 → C3. Reclassified from Theorem T4: this result is a corollary of T5, not an independent theorem.

VI. Universal Structure vs. Contingent Parameters

Remark R1 — Universal Structure vs. Universe-Contingent Parameters

Universal (invariant across all instantiations): AX-B1 (binary distinction — logical, not physical). MP-1 (methodological commitment). T0 (p=2 derived). T1, T2, T3, T5, C1, C2, C3 (all topological results). Structural role of ε_0.

Universe-contingent (varies across instantiations): Numerical value of ε_0 (determined by physical constants). Physical predictions invoking ε_0 numerically.

Consequence: The Zero Paradox is a universal ontology of state emergence, not a physical theory of our universe specifically.

VII. Boundary Conditions for ZP-D and ZP-E

Export	Status	Receiving Document
AX-B1	Axiom	ZP-E: foundational axiom
MP-1	Principle	ZP-E: bridge between ontological and representational binary
T0: $p = 2$	Derived from AX-B1 + MP-1	ZP-D: domain of T is Q_2
Q_2 with 2-adic metric (D1, D2)	Defined	ZP-D: topological domain of T

Export	Status	Receiving Document
T1: Ultrametric	Derived	ZP-D: non-Archimedean structure
T2: Clopen balls	Derived	ZP-D: topological isolation maps to orthogonality in H
T3: Topological isolation of 0	Derived	ZP-E: grounds ontological claim about the Snap
T5: Total disconnectedness	Derived	ZP-E: supports C3
C3: Snap topologically irreversible	Derived — Corollary of T5	ZP-E: cross-framework irreversibility
ε_0 (D5)	Defined — contingent	ZP-E: Snap threshold; value depends on instantiation
R1: Universal vs. contingent	Remark	ZP-E: framework is instantiation-independent

VIII. Validation Status

Component	Status / Notes
AX-B1	Axiom — explicit; load-bearing premise
MP-1	Principle — explicit bridge; resolves reviewer gap in T0
T0: p = 2	Valid — Derived given MP-1 (design commitment). MP-1 encodes the minimality choice; T0 follows. OQ-B1 closed.
D1: 2-adic absolute value	Valid — standard definition
D2: 2-adic metric	Valid — follows from D1
T1: Strong triangle inequality	Valid — Derived
C1: All triangles isosceles	Valid — Corollary of T1
T2: Every ball is clopen	Valid — Derived from T1
C2: Disjoint balls do not communicate	Valid — Derived from T2 only; forward citation to T5 removed
T3: Topological isolation of 0	Valid — Derived from D1 and D2
T5: Q_2 totally disconnected	Valid — Derived from T2; proven before C3
C3: Snap topologically irreversible	Valid — Corollary of T5; reclassified from Theorem T4
D5: ε_0	Valid — Defined; structural role universal; value contingent
R1	Valid — Remark; ontological scope clarified