

# THE ZERO PARADOX

*ZP-B: p-Adic Topology*

*Version 1.5 | May 2026*

*Supersedes v1.4 | v1.5: T3 ZP interpretation bullets labelled explicitly — pure topology and ZP-specific framing now separated with "ZP Interpretation:" prefix.  
v1.4: T0 reframed: derived given MP-1 (design commitment); MP-1 acknowledged as load-bearing choice.*

This document is self-contained within p-adic analysis and topology. No abstract algebra from ZP-A, no probability, and no Hilbert space is imported. Cross-framework connections are deferred to ZP-D and ZP-E.

Illustrated Companion: A paired ZP-B Illustrated Companion provides concrete examples and visual intuitions for the results here. Examples are kept separate from the formal layers to distinguish illustrative material from proofs.

Version 1.4 change: T0 status updated from "DERIVED from AX-B1 and MP-1" to "DERIVED given MP-1 (design commitment)". MP-1 is the load-bearing design choice; T0 is a valid derivation given MP-1. OQ-B1 closed given MP-1.

Version 1.3 change: Theorem/Proposition hierarchy applied. T1 and T2 relabelled Proposition. T5 relabelled Proposition. T0 and T3 retain Theorem labels.

Version 1.2 changes: T0 strengthened with MP-1; C2 fixed to derive from T2 only; T4 reclassified as C3 (corollary of T5).

## I. The Foundational Distinction

### 1.1 The Binary Existence Axiom

Axiom AX-B1 — Binary Existence
The foundational distinction of the Zero Paradox framework is binary: a state either exists or it does not. There is no third option at this level.
0 — non-existence (the Null State, corresponding to $\perp$ in ZP-A)
1 — existence (the First Atomic State, the minimal non-zero element)
Status: AXIOM. This is the only non-topological commitment in ZP-B. It precedes p-adic analysis and is the premise from which the field selection is derived.
Scope: AX-B1 asserts the structure of the ontological distinction, not its physical realisation. It is invariant across all instantiations.

### 1.2 The Minimality Principle

## Principle MP-1 — Minimality of Representation

The representational base of the framework must be the minimum base capable of encoding the ontological distinction of AX-B1 without redundancy and without information loss.

Redundancy: a base  $p > \text{minimum}$  introduces representational states with no ontological counterpart, violating parsimony.

Information loss: a base  $p < \text{minimum}$  cannot distinguish all ontologically distinct states, violating faithfulness.

Status: PRINCIPLE — methodological commitment. Given AX-B1 (two ontological states) and MP-1 (minimum sufficient base), the representational base is uniquely determined as 2.

### 1.3 Derivation of $p = 2$

#### Theorem T0 — $p = 2$ is the Unique Minimum Sufficient Representational Base

Given AX-B1 and MP-1, the  $p$ -adic field appropriate for the Zero Paradox framework is  $Q_p$ .

Note on MP-1: MP-1 is a design commitment — the choice to use the minimum sufficient base. Any  $Q_p$  for prime  $p \geq 2$  contains elements 0 and 1 capable of representing the AX-B1 distinction; the choice of minimum base is what MP-1 encodes. T0 is a valid derivation from AX-B1 and MP-1, but MP-1 is the load-bearing design choice, not a mathematical necessity. OQ-B1 is closed given MP-1.

Proof:

Step 1 — AX-B1 establishes exactly two ontological states: non-existence (0) and existence (1).

Step 2 — A  $p$ -adic field  $Q_p$  uses coefficients from  $\{0, 1, \dots, p-1\}$ . The minimum base  $p$  capable of representing exactly two distinct values without redundancy is  $p = 2$ , with coefficient set  $\{0, 1\}$ . One coefficient per ontological state; no unused coefficients.

Step 3 —  $p = 1$  has only one coefficient value  $\{0\}$ . Cannot distinguish existence from non-existence. Fails faithfulness (MP-1).

Step 4 —  $p > 2$ : coefficient set  $\{0, \dots, p-1\}$  contains values with no ontological counterpart. Violates no-redundancy condition of MP-1.

Step 5 —  $p = 2$  is the unique prime satisfying both conditions simultaneously.

Step 6 — The binary branching at every level of  $Q_2$ 's ball structure reflects the eventual binary resolution of any representational complexity.

Therefore  $p = 2$ . Status: DERIVED given MP-1 (design commitment). OQ-B1 closed. ✓

## II. The 2-Adic Field

### 2.1 The 2-Adic Absolute Value

Fix  $p = 2$  (derived in T0). Every non-zero rational  $q \in \mathbb{Q}$  can be written uniquely as  $q = 2^v \cdot (a/b)$  where  $v \in \mathbb{Z}$  and  $a, b$  are integers not divisible by 2. The integer  $v$  is the 2-adic valuation  $v_2(q)$ . By convention,  $v_2(0) =$

$+\infty$ .

### Definition D1 — 2-Adic Absolute Value

For  $q \in \mathbb{Q}$ :  $|q|_2 = 2^{-v_2(q)}$  for  $q \neq 0$ ;  $|0|_2 = 0$

Elements with high powers of 2 are considered small under  $|\cdot|_2$ . Elements with no factor of 2 have  $|\cdot|_2 = 1$ .

## 2.2 The 2-Adic Field $\mathbb{Q}_2$

$\mathbb{Q}_2$  is the completion of  $\mathbb{Q}$  under the metric induced by  $|\cdot|_2$ . Elements of  $\mathbb{Q}_2$  are formal power series in 2:  $x = \sum_{n=v}^{\infty} a_n \cdot 2^n$  where  $a_n \in \{0,1\}$ . The coefficients  $a_n \in \{0,1\}$  are precisely the binary values of AX-B1.

## 2.3 The Minimum Viable Deviation $\epsilon_0$

### Definition D5 — Minimum Viable Deviation $\epsilon_0$

$\epsilon_0 = 2^k$  for some integer  $k$ , where  $k$  is the maximum valuation accessible in the instantiation.

Structural role (universal):  $\epsilon_0$  is always the first element crossed by the Snap. Fixed by the structure of  $\mathbb{Q}_2$  and AX-B1.

Numerical value (contingent): determined by physical constants of the instantiation. Planck-scale quantities in our universe.

Status: DEFINED — universe-contingent parameter.

## III. The Ultrametric

### Definition D2 — 2-Adic Metric

For  $x, y \in \mathbb{Q}_2$ :  $d(x, y) = |x - y|_2$

### Proposition T1 — Strong Triangle Inequality (Ultrametric)

For all  $x, y, z \in \mathbb{Q}_2$ :  $d(x, z) \leq \max(d(x, y), d(y, z))$

Proof: Write  $x - z = (x - y) + (y - z)$ . The ultrametric property of  $v_2$  gives  $v_2(a+b) \geq \min(v_2(a), v_2(b))$ , from which  $|a+b|_2 \leq \max(|a|_2, |b|_2)$ . Apply with  $a = x-y$  and  $b = y-z$ . ✓

### Corollary C1 — All Triangles are Isosceles

If  $d(x,y) \neq d(y,z)$ , then  $d(x,z) = \max(d(x,y), d(y,z))$ .

Proof: Suppose  $d(x,y) < d(y,z)$ . By T1,  $d(x,z) \leq d(y,z)$ . Also  $d(y,z) \leq \max(d(y,x), d(x,z)) = d(x,z)$  since  $d(x,y) < d(y,z)$ . Therefore  $d(x,z) = d(y,z)$ . ✓

## 3.2 Clopen Ball Structure

### Definition D3 — Ball in $Q_2$

$B(a, r) = \{ x \in Q_2 : d(x, a) \leq r \}$  (closed ball)

$B^\circ(a, r) = \{ x \in Q_2 : d(x, a) < r \}$  (open ball)

### Proposition T2 — Every Ball is Clopen

In  $Q_2$ , every closed ball is also open and every open ball is also closed.

Proof (closed ball is open): Let  $y \in B(a, r)$ . For any  $z \in B(y, r)$ , T1 gives  $d(z, a) \leq \max(d(z, y), d(y, a)) \leq r$ . So  $B(y, r) \subseteq B(a, r)$ . Every point is an interior point. ✓

Proof (open ball is closed): Let  $(x_n) \rightarrow x$  with all  $x_n \in B^\circ(a, r)$ . Ball radii in  $Q_2$  are discrete (powers of 2), so  $d(x, a) < r$  holds in the limit. Thus  $x \in B^\circ(a, r)$ . ✓

### Corollary C2 — Disjoint Balls Do Not Communicate [v1.2: derived from T2 only]

If  $B(a, r)$  and  $B(b, r)$  are disjoint ( $d(a, b) > r$ ), then no continuous path exists from any point in  $B(a, r)$  to any point in  $B(b, r)$ .

Proof: By T2,  $B(a, r)$  is clopen in  $Q_2$ . Any continuous  $f: [0, 1] \rightarrow Q_2$  with  $f(0) \in B(a, r)$  and  $f(1) \in B(b, r)$  would require  $f$  to map the connected set  $[0, 1]$  onto a subset intersecting both  $B(a, r)$  and its clopen complement. The preimage of a clopen set under a continuous function is clopen in  $[0, 1]$ . Since  $[0, 1]$  is connected, the preimage is either empty or all of  $[0, 1]$ . It cannot be all of  $[0, 1]$  (since  $f(1) \notin B(a, r)$ ) and cannot be empty (since  $f(0) \in B(a, r)$ ). Contradiction. ✓

## IV. Topological Isolation of Zero

### Theorem T3 — Topological Isolation of 0

For any  $r = 2^{-k}$ , the ball  $B(0, r) = \{ x \in Q_2 : v_2(x) \geq k \}$ . Any  $x$  outside this ball has  $d(0, x) \geq 2^{-k+1} > r$ .  $B(0, r)$  and its complement are separated by a gap of at least  $2^{-k}$ .

ZP Interpretation: this discrete jump across a clopen boundary is the topological expression of the Binary Snap (defined in ZP-E).

Relationship to  $\varepsilon_0$ :  $\varepsilon_0 = 2^k$  is the smallest non-zero element outside the tightest ball around 0. ZP Interpretation: this is the gap the Binary Snap crosses. ✓

## V. Topological Structure of $Q_2$

### 5.1 Total Disconnectedness — proven before C3

#### Proposition T5 — $Q_2$ is Totally Disconnected

The only connected subsets of  $Q_2$  are singletons.

### Proposition T5 — $Q_2$ is Totally Disconnected

Proof: Let  $S \subseteq Q_2$  contain two distinct points  $a, b$  with  $d(a,b) = r > 0$ . Choose  $s$  with  $0 < s < r$ . By T2,  $B(a,s)$  is clopen. Then  $S = [S \cap B(a,s)] \cup [S \setminus B(a,s)]$  is a separation of  $S$  into two disjoint non-empty clopen sets. Therefore  $S$  is not connected. Since  $S$  was arbitrary, the only connected subsets are singletons. ✓

## 5.2 Topological Irreversibility of the Snap

### Definition D4 — Topological Irreversibility

A transition from  $a$  to  $b$  in a topological space  $X$  is topologically irreversible if there exists no continuous path  $\gamma: [0,1] \rightarrow X$  with  $\gamma(0) = b$  and  $\gamma(1) = a$ .

### Corollary C3 — The Snap is Topologically Irreversible [reclassified from T4 in v1.1]

Let  $x \in Q_2$  with  $x \neq 0$ . There exists no continuous path  $\gamma: [0,1] \rightarrow Q_2$  with  $\gamma(0) = x$  and  $\gamma(1) = 0$ .

Proof: By T5,  $Q_2$  is totally disconnected. A continuous path  $\gamma$  with  $\gamma(0) = x \neq 0$  and  $\gamma(1) = 0$  would require  $\gamma([0,1])$  to be a connected subset of  $Q_2$  containing two distinct points. By T5, no such connected subset exists. ✓

Derivation chain: T1 → T2 → T5 → C3. Reclassified from Theorem T4: this result is a corollary of T5, not an independent theorem.

## VI. Universal Structure vs. Contingent Parameters

### Remark R1 — Universal Structure vs. Universe-Contingent Parameters

Universal (invariant across all instantiations): AX-B1 (binary distinction — logical, not physical). MP-1 (methodological commitment). T0 ( $p=2$  derived). T1, T2, T3, T5, C1, C2, C3 (all topological results). Structural role of  $\epsilon_0$ .

Universe-contingent (varies across instantiations): Numerical value of  $\epsilon_0$  (determined by physical constants). Physical predictions invoking  $\epsilon_0$  numerically.

Consequence: The Zero Paradox is a universal ontology of state emergence, not a physical theory of our universe specifically.

## VII. Boundary Conditions for ZP-D and ZP-E

Export	Status	Receiving Document
AX-B1	Axiom	ZP-E: foundational axiom
MP-1	Principle	ZP-E: bridge between ontological and representational binary
T0: $p = 2$	Derived from AX-B1 + MP-1	ZP-D: domain of T is $Q_2$
$Q_2$ with 2-adic metric (D1, D2)	Defined	ZP-D: topological domain of T

Export	Status	Receiving Document
T1: Ultrametric	Derived	ZP-D: non-Archimedean structure
T2: Clopen balls	Derived	ZP-D: topological isolation maps to orthogonality in H
T3: Topological isolation of 0	Derived	ZP-E: grounds ontological claim about the Snap
T5: Total disconnectedness	Derived	ZP-E: supports C3
C3: Snap topologically irreversible	Derived — Corollary of T5	ZP-E: cross-framework irreversibility
$\epsilon_0$ (D5)	Defined — contingent	ZP-E: Snap threshold; value depends on instantiation
R1: Universal vs. contingent	Remark	ZP-E: framework is instantiation-independent

## VIII. Validation Status

Component	Status / Notes
AX-B1	Axiom — explicit; load-bearing premise
MP-1	Principle — explicit bridge; resolves reviewer gap in T0
T0: $p = 2$	Valid — Derived given MP-1 (design commitment). MP-1 encodes the minimality choice; T0 follows. OQ-B1 closed.
D1: 2-adic absolute value	Valid — standard definition
D2: 2-adic metric	Valid — follows from D1
T1: Strong triangle inequality	Valid — Derived
C1: All triangles isosceles	Valid — Corollary of T1
T2: Every ball is clopen	Valid — Derived from T1
C2: Disjoint balls do not communicate	Valid — Derived from T2 only; forward citation to T5 removed
T3: Topological isolation of 0	Valid — Derived from D1 and D2
T5: $\mathbb{Q}_2$ totally disconnected	Valid — Derived from T2; proven before C3
C3: Snap topologically irreversible	Valid — Corollary of T5; reclassified from Theorem T4
D5: $\epsilon_0$	Valid — Defined; structural role universal; value contingent
R1	Valid — Remark; ontological scope clarified