THE ZERO PARADOX

ZP-B: p-Adic Topology

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This document is self-contained within p-adic analysis and topology. No abstract algebra from ZP-A, no probability, and no Hilbert space is imported. Cross-framework connections are deferred to ZP-D and ZP-E.

Illustrated Companion: A paired ZP-B Illustrated Companion provides concrete examples and visual intuitions for the results here. Examples are kept separate from the formal layers to distinguish illustrative material from proofs.

I. The Foundational Distinction

1.1 The Binary Existence Axiom

Axiom AX-B1 — Binary Existence
The foundational distinction of the Zero Paradox framework is binary: a state either exists or it does not. There is no third option at this level.
0 — non-existence (the Null State, corresponding to ⊥ in ZP-A)
1 — existence (the First Atomic State, the minimal non-zero element)
Status: AXIOM. This is the only non-topological commitment in ZP-B. It precedes p-adic analysis and is the premise from which the field selection is derived.
Scope: AX-B1 asserts the structure of the ontological distinction, not its physical realisation. It is invariant across all instantiations.

Note on set membership: both 0 and 1 are fully present as mathematical objects — neither is absent from the formal structure. The set is not a collection of things that happen to exist; it is a model of an ontological situation. 0 names the null condition: the state in which nothing has been instantiated. 1 names the first non-null condition: the state in which something has. Existence in AX-B1 is a property the states represent, not a statement about set membership.

Lean encoding: AX-B1 is encoded as OntologicalStates — a free inductive type with two named constructors: .null (non-existence) and .exist (existence). This avoids tying the null state to any numeric convention such as ℕ's 0. Distinctness (null ≠ exist) is verified by decide via deriving DecidableEq — no classical axioms required. Lean identifiers do not appear in the companion documents, which are Lean-free by design.

1.2 The Minimality Principle

Principle MP-1 — Minimality of Representation
The representational base of the framework must be the minimum base capable of encoding the ontological distinction of AX-B1 without redundancy and without information loss.
Redundancy: a base $p >$ minimum introduces representational states with no ontological counterpart, violating parsimony.
Information loss: a base $p <$ minimum cannot distinguish all ontologically distinct states, violating faithfulness.
Status: PRINCIPLE — methodological commitment. Given AX-B1 (two ontological states) and MP-1 (minimum sufficient base), the representational base is uniquely determined as 2.

1.3 Derivation of p = 2

Theorem T0 — p = 2 is the Unique Minimum Sufficient Representational Base
Given AX-B1 and MP-1, the p-adic field appropriate for the Zero Paradox framework is Q_2.
Note on MP-1: MP-1 is a design commitment — the choice to use the minimum sufficient base. Any Q_p for prime $p \geq 2$ contains elements 0 and 1 capable of representing the AX-B1 distinction; the choice of minimum base is what MP-1 encodes. T0 is a valid derivation from AX-B1 and MP-1, but MP-1 is the load-bearing design choice, not a mathematical necessity. OQ-B1 is closed given MP-1.
Proof:
Step 1 — AX-B1 establishes exactly two ontological states: non-existence (0) and existence (1).
Step 2 — A p-adic field Q_p uses coefficients from $\{0, 1, \ldots, p-1\}$. The minimum base p capable of representing exactly two distinct values without redundancy is $p = 2$, with coefficient set $\{0, 1\}$. One coefficient per ontological state; no unused coefficients.
Step 3 — $p = 1$ has only one coefficient value $\{0\}$. Cannot distinguish existence from non-existence. Fails faithfulness (MP-1).
Step 4 — $p > 2$: coefficient set $\{0, \ldots, p-1\}$ contains values with no ontological counterpart. Violates no-redundancy condition of MP-1.
Step 5 — $p = 2$ is the unique prime satisfying both conditions simultaneously.
Step 6 — The binary branching at every level of Q_2's ball structure reflects the eventual binary resolution of any representational complexity.
Therefore $p = 2$. Status: DERIVED given MP-1 (design commitment). OQ-B1 closed. ✓

II. The 2-Adic Field

2.1 The 2-Adic Absolute Value

Fix $p = 2$ (derived in T0). Every non-zero rational $q \in \mathbb{Q}$ can be written uniquely as $q = 2^v \cdot (a/b)$ where $v \in \mathbb{Z}$ and a, b are integers not divisible by 2. The integer v is the 2-adic valuation $v_2(q)$. By convention, $v_2(0) =$

$+\infty$.

Definition D1 — 2-Adic Absolute Value

For $q \in \mathbb{Q}$: $|q|_2 = 2^{-v_2(q)}$ for $q \neq 0$; $|0|_2 = 0$

Elements with high powers of 2 are considered small under $|\cdot|_2$. Elements with no factor of 2 have $|\cdot|_2 = 1$.

2.2 The 2-Adic Field Q_2

Q_2 is the completion of $\mathbb{Q}$ under the metric induced by $|\cdot|_2$. Elements of Q_2 are formal power series in 2: $x = \sum_{n=v}^{\infty} a_n \cdot 2^n$ where $a_n \in \{0,1\}$. The coefficients $a_n \in \{0,1\}$ are precisely the binary values of AX-B1.

2.3 The Minimum Viable Deviation ε_0

Definition D5 — Minimum Viable Deviation ε_0

$\varepsilon_0 = 2^k$ for some integer k, where k is the maximum valuation accessible in the instantiation.

Structural role (universal): ε_0 is always the first element crossed by the Snap. Fixed by the structure of Q_2 and AX-B1.

Numerical value (contingent): determined by physical constants of the instantiation. Planck-scale quantities in our universe.

Status: DEFINED — universe-contingent parameter.

III. The Ultrametric

Definition D2 — 2-Adic Metric

For $x, y \in Q_2$: $d(x, y) = |x - y|_2$

Proposition T1 — Strong Triangle Inequality (Ultrametric)

For all $x, y, z \in Q_2$: $d(x, z) \leq \max(d(x, y), d(y, z))$

Proof: Write $x - z = (x - y) + (y - z)$. The ultrametric property of v_2 gives $v_2(a+b) \geq \min(v_2(a), v_2(b))$, from which $|a+b|_2 \leq \max(|a|_2, |b|_2)$. Apply with $a = x-y$ and $b = y-z$. ✓

Corollary C1 — All Triangles are Isosceles

If $d(x,y) \neq d(y,z)$, then $d(x,z) = \max(d(x,y), d(y,z))$.

Proof: Suppose $d(x,y) < d(y,z)$. By T1, $d(x,z) \leq d(y,z)$. Also $d(y,z) \leq \max(d(y,x), d(x,z)) = d(x,z)$ since $d(x,y) < d(y,z)$. Therefore $d(x,z) = d(y,z)$. ✓

3.2 Clopen Ball Structure

Definition D3 — Ball in Q_2

$B(a, r) = \{ x \in Q_2 : d(x, a) \leq r \}$ (closed ball)

$B°(a, r) = \{ x \in Q_2 : d(x, a) < r \}$ (open ball)

Proposition T2 — Every Ball is Clopen

In Q_2, every closed ball is also open and every open ball is also closed.

Proof (closed ball is open): Let $y \in B(a, r)$. For any $z \in B(y, r)$, T1 gives $d(z, a) \leq \max(d(z,y), d(y,a)) \leq r$. So $B(y,r) \subseteq B(a,r)$. Every point is an interior point. ✓

Proof (open ball is closed): Let $(x_n) \to x$ with all $x_n \in B°(a,r)$. Ball radii in Q_2 are discrete (powers of 2), so $d(x,a) < r$ holds in the limit. Thus $x \in B°(a,r)$. ✓

Corollary C2 — Disjoint Balls Do Not Communicate [v1.2: derived from T2 only]

If $B(a, r)$ and $B(b, r)$ are disjoint ($d(a,b) > r$), then no continuous path exists from any point in $B(a, r)$ to any point in $B(b, r)$.

Proof: By T2, $B(a, r)$ is clopen in Q_2. Any continuous $f: [0,1] \to Q_2$ with $f(0) \in B(a,r)$ and $f(1) \in B(b,r)$ would require f to map the connected set $[0,1]$ onto a subset intersecting both $B(a,r)$ and its clopen complement. The preimage of a clopen set under a continuous function is clopen in $[0,1]$. Since $[0,1]$ is connected, the preimage is either empty or all of $[0,1]$. It cannot be all of $[0,1]$ (since $f(1) \notin B(a,r)$) and cannot be empty (since $f(0) \in B(a,r)$). Contradiction. ✓

IV. Topological Isolation of Zero

Theorem T3 — Topological Isolation of 0

For any $r = 2^{-k}$, the ball $B(0,r) = \{ x \in Q_2 : v_2(x) \geq k \}$. Any x outside this ball has $d(0,x) \geq 2^{-k+1} > r$. $B(0,r)$ and its complement are separated by a gap of at least 2^{-k}.

ZP Interpretation: this discrete jump across a clopen boundary is the topological expression of the Binary Snap (defined in ZP-E).

Relationship to ε_0: $\varepsilon_0 = 2^k$ is the smallest non-zero element outside the tightest ball around 0. ZP Interpretation: this is the gap the Binary Snap crosses. ✓

V. Topological Structure of Q_2

5.1 Total Disconnectedness — proven before C3

Proposition T5 — Q_2 is Totally Disconnected

The only connected subsets of Q_2 are singletons.

Proposition T5 — Q_2 is Totally Disconnected

Proof: Let $S \subseteq Q_2$ contain two distinct points a, b with $d(a,b) = r > 0$. Choose s with $0 < s < r$. By T2, $B(a,s)$ is clopen. Then $S = [S \cap B(a,s)] \cup [S \setminus B(a,s)]$ is a separation of S into two disjoint non-empty clopen sets. Therefore S is not connected. Since S was arbitrary, the only connected subsets are singletons. ✓

5.2 Topological Irreversibility of the Snap

Definition D4 — Topological Irreversibility

A transition from a to b in a topological space X is topologically irreversible if there exists no continuous path $\gamma: [0,1] \to X$ with $\gamma(0) = b$ and $\gamma(1) = a$.

Corollary C3 — The Snap is Topologically Irreversible [reclassified from T4 in v1.1]

Let $x \in Q_2$ with $x \neq 0$. There exists no continuous path $\gamma: [0,1] \to Q_2$ with $\gamma(0) = x$ and $\gamma(1) = 0$.

Proof: By T5, Q_2 is totally disconnected. A continuous path γ with $\gamma(0) = x \neq 0$ and $\gamma(1) = 0$ would require $\gamma([0,1])$ to be a connected subset of Q_2 containing two distinct points. By T5, no such connected subset exists. ✓

Derivation chain: T1 → T2 → T5 → C3. Reclassified from Theorem T4: this result is a corollary of T5, not an independent theorem.

VI. Universal Structure vs. Contingent Parameters

Remark R1 — Universal Structure vs. Universe-Contingent Parameters

Universal (invariant across all instantiations): AX-B1 (binary distinction — logical, not physical). MP-1 (methodological commitment). T0 ($p=2$ derived). T1, T2, T3, T5, C1, C2, C3 (all topological results). Structural role of ε_0.

Universe-contingent (varies across instantiations): Numerical value of ε_0 (determined by physical constants). Physical predictions invoking ε_0 numerically.

Consequence: The Zero Paradox is a universal ontology of state emergence, not a physical theory of our universe specifically.

VII. Boundary Conditions for ZP-D and ZP-E

Export	Status	Receiving Document
AX-B1	Axiom	ZP-E: foundational axiom
MP-1	Principle	ZP-E: bridge between ontological and representational binary
T0: $p = 2$	Derived from AX-B1 + MP-1	ZP-D: domain of T is Q_2
Q_2 with 2-adic metric (D1, D2)	Defined	ZP-D: topological domain of T

Export	Status	Receiving Document
T1: Ultrametric	Derived	ZP-D: non-Archimedean structure
T2: Clopen balls	Derived	ZP-D: topological isolation maps to orthogonality in H
T3: Topological isolation of 0	Derived	ZP-E: grounds ontological claim about the Snap
T5: Total disconnectedness	Derived	ZP-E: supports C3
C3: Snap topologically irreversible	Derived — Corollary of T5	ZP-E: cross-framework irreversibility
ε_0 (D5)	Defined — contingent	ZP-E: Snap threshold; value depends on instantiation
R1: Universal vs. contingent	Remark	ZP-E: framework is instantiation-independent

VIII. Validation Status

Component	Status / Notes
AX-B1	Axiom — explicit; load-bearing premise
MP-1	Principle — explicit bridge; resolves reviewer gap in T0
T0: $p = 2$	Valid — Derived given MP-1 (design commitment). MP-1 encodes the minimality choice; T0 follows. OQ-B1 closed.
D1: 2-adic absolute value	Valid — standard definition
D2: 2-adic metric	Valid — follows from D1
T1: Strong triangle inequality	Valid — Derived
C1: All triangles isosceles	Valid — Corollary of T1
T2: Every ball is clopen	Valid — Derived from T1
C2: Disjoint balls do not communicate	Valid — Derived from T2 only; forward citation to T5 removed
T3: Topological isolation of 0	Valid — Derived from D1 and D2
T5: Q_2 totally disconnected	Valid — Derived from T2; proven before C3
C3: Snap topologically irreversible	Valid — Corollary of T5; reclassified from Theorem T4
D5: ε_0	Valid — Defined; structural role universal; value contingent
R1	Valid — Remark; ontological scope clarified