THE ZERO PARADOX

ZP-D: State Layer (Hilbert Space)

Version 1.5 | April 2026

Supersedes v1.4 | D1 updated: n = 2 established as foundational minimum (binary existence/non-existence); further dimensions derived, not foundational; core claims T4 and T5 grounded at n = 2

v1.4: T5 proof corrected — ball-boundary argument replaces D2(v) citation

This document operates within functional analysis. It imports from ZP-A and ZP-B and constructs the Hilbert space state layer on top of them. No information theory from ZP-C is imported. Cross-framework synthesis is deferred to ZP-E.

Illustrated Companion: A paired ZP-D Illustrated Companion provides concrete examples and visual intuitions for the results here. Examples are kept separate from the formal layers to distinguish illustrative material from proofs.

Version 1.4 change: T5 proof corrected. Prior proof cited D2(v) (global lower bound $\|T(x)\| \geq \|T(0)\|$) as justification for sequence monotonicity — this does not follow. Correct proof uses ball-boundary argument: norm is non-decreasing because T maps each clopen ball to a single basis vector; crossing a ball boundary adds a component (strict increase); staying within a ball gives equality. T5 is renamed "Non-Decreasing Norms" to reflect this precisely.

Version 1.3 change: Theorem/Proposition hierarchy applied. T3 relabelled Proposition. T5 relabelled Proposition. T2 and T4 retain Theorem labels.

Version 1.2 change: Theorem T1 is reclassified as Design Principle DP-1. Orthogonality is a design commitment — well-motivated and explicit — but chosen, not derived.

I. Imported Structure

1.1 From ZP-A: Algebraic Structure of States

Import I-A — From ZP-A: Lattice Algebra
$(L, \vee, \bot)$: join-semilattice with bottom. Axioms A1–A4.
$\leq$ partial order: $x \leq y :\iff x \vee y = y$ (D1, T1).
$\bot$ is the global minimum: $\bot \leq x$ for all $x \in L$ (T2).
State transitions are joins: $f(x) = x \vee \alpha$ (D2).
State sequences are monotone: $S_n \leq S_{n+1}$ (T3).
CC-1: $S_0 = \bot$ — Conditional Claim; modelling commitment.

1.2 From ZP-B: Topological Domain

Import I-B — From ZP-B: p-Adic Topology
AX-B1: Binary Existence Axiom.
MP-1: Minimality Principle.
T0: $p = 2$ derived from AX-B1 and MP-1.
Q_2 with 2-adic metric d (D1, D2).
T1: Ultrametric (strong triangle inequality).
T2: Every ball is clopen.
T3: Topological isolation of 0.
T5: Q_2 is totally disconnected.
C3: Snap is topologically irreversible (corollary of T5).
ε_0: Minimum viable deviation, universe-contingent parameter (D5).

II. The Hilbert Space State Layer

Definition D1 — State Layer H
$H = \mathbb{C}^n$ is a complex Hilbert space with orthonormal basis $\{e_0, e_1, \ldots\}$.
The foundational minimum is $n = 2$, corresponding to the two ontological states of the framework: ⊥ (null state, mapped to e_0) and ε_0 (first atomic state, mapped to e_1). These two orthogonal vectors express the binary existence/non-existence distinction that is the framework's central object. All further states are derived from this pair as joins in $(L, \vee, \perp)$ — they require no additional foundational dimension.
The framework's core claims (T4: snap produces orthogonal shift; T5: monotone norms) are established at $n = 2$. Extensions to higher n are consistent and natural: for level-k approximations using the clopen ball partition of Q_2 at depth k, $n = 2^k$. No foundational claim of the framework requires $n > 2$.

Remark R1 — Decoupling of Topological and State Layers
Q_2 and H are categorically distinct structures. Q_2 is a topological field; H is a Hilbert space over $\mathbb{C}$. They share no operations.
The transition operator $T: Q_2 \to H$ is the only bridge. It is constructed explicitly in T2.
No operation in H is assumed to inherit a topological property of Q_2 without proof. Every cross-layer claim must go through T.

III. The Transition Operator $T: Q_2 \to H$

3.1 The Design Commitment — Orthogonality

Design Principle DP-1 — Orthogonality as the Representation of Topological Isolation [reclassified from T1 in v1.1]
Topological isolation in Q_2 (T3: 0 is isolated; clopen balls are mutually separated) is represented in H by orthogonality: elements that are topologically isolated in Q_2 map to orthogonal vectors in H.
Motivation: Orthogonality in H is the natural algebraic analogue of topological separation. $\langle e_i, e_j \rangle = 0$ for $i \neq j$; two clopen balls are maximally distinct in the topological sense.
Status: DESIGN PRINCIPLE — DP-1 is chosen, not derived. It is the natural and consistent choice, stated explicitly. T4 and T5 below depend on DP-1 as a premise.

3.2 The Construction Target

Definition D2 — Transition Operator T (Requirements)
T: $Q_2 \to H$ must satisfy:
(i) $T(0) = e_0$ (null state maps to the designated base vector)
(ii) $T(\varepsilon_0) = e_1$ (minimum deviation maps to the first non-base vector)
(iii) T is injective on the clopen ball partition
(iv) If x and y are in disjoint clopen balls, then $\langle T(x), T(y) \rangle = 0$ (DP-1)
(v) $\|T(x)\| \geq \|T(0)\|$ for all x (norm-increasing: additive ontology preserved)

3.3 Existence of T

Theorem T2 — Existence of T (Basis Assignment)
There exists a function T: $Q_2 \to H$ satisfying all five requirements of D2.
Proof: Construct T by basis assignment. The clopen ball partition of Q_2 at level k consists of 2^k disjoint clopen balls. Assign each ball to a distinct basis vector of H. $T(x) = e_i$ where i is the index of the ball containing x.
(i) 0 maps to e_0 by assignment. ✓ (ii) ε_0 maps to e_1 by assignment. ✓ (iii) Distinct balls → distinct basis vectors. ✓ (iv) Disjoint balls → orthogonal basis vectors. ✓ (v) All basis vectors have norm $1 \geq \|e_0\| = 1$. ✓

3.4 Uniqueness of T

Proposition T3 — Uniqueness of T up to Unitary Equivalence
Any two operators T, T': $Q_2 \to H$ satisfying D2 are related by a unitary transformation U: H → H such that $T' = U \circ T$.
Proof: T and T' both assign e_0 to the image of 0, which is the unique additive identity (A4). The ball structure of Q_2 is fixed; only the labelling of basis vectors varies. A unitary map U taking T(0) to T'(0) and preserving orthogonality relations defines the equivalence. ✓

Remark R2 — What T Is Not

T is not a ring homomorphism. Q_2 has field operations; H does not. T does not preserve addition or multiplication from Q_2.

T is not a topological embedding. The topology of H is the norm topology; the topology of Q_2 is the 2-adic ultrametric. T is a structure-preserving assignment: ontological distinctions (topological isolation in Q_2) map to algebraic distinctions (orthogonality in H), as specified by DP-1.

IV. The Binary Snap in H

Theorem T4 — Snap Produces Orthogonal Shift in H

The Binary Snap $0 \to \varepsilon_0$ in Q_2 maps to an orthogonal shift in H: $T(0) = e_0$ and $T(\varepsilon_0) = e_1$, and $\langle e_0, e_1 \rangle = 0$.

Proof: By D2(i), $T(0) = e_0$. By D2(ii), $T(\varepsilon_0) = e_1$. Since 0 and ε_0 are in disjoint clopen balls of Q_2 (T3), D2(iv) and DP-1 give $\langle T(0), T(\varepsilon_0)\rangle = \langle e_0, e_1\rangle = 0$. ✓

Status: Derived — unconditional theorem given DP-1.

Proposition T5 — Monotone Sequences Map to Non-Decreasing Norms

Let (S_n) be a monotone state sequence in L (ZP-A T3). Then $\|T(S_n)\| \leq \|T(S_{n+1})\|$ for all n.

Proof: By ZP-A T3, $S_n \leq S_{n+1}$. T maps each clopen ball of Q_2 to a single basis vector (D2). If S_n and S_{n+1} lie in different clopen balls, $T(S_{n+1})$ carries an additional basis component, giving $\|T(S_{n+1})\| > \|T(S_n)\|$. If they lie in the same ball, $T(S_n) = T(S_{n+1})$, giving equality. In both cases $\|T(S_n)\| \leq \|T(S_{n+1})\|$. Note: D2(v) gives a global lower bound $\|T(x)\| \geq \|T(0)\|$ — this does not imply sequence monotonicity; the ball-boundary argument above is the correct proof. ✓

V. Open Items Register for ZP-D v1.5

Item	Status	Description
DP-1: Orthogonality commitment	Design Principle — explicit	Reclassified from Theorem T1. Orthogonality is chosen, not derived. Content unchanged.
T2: Existence of T	Closed	Basis assignment construction. All five requirements verified.
T3: Uniqueness of T	Closed	Unique up to unitary equivalence.
T4: Snap → orthogonal shift	Closed — unconditional	Proven from T2 and ZP-B T3. Depends on DP-1 as premise.
T5: Monotone norms	Closed — unconditional	Proven from T2 and ZP-A T3.

VI. Validation Status

Component	Status / Notes
$H = \mathbb{C}^n$ (D1)	Valid — Defined; foundational minimum $n = 2$ (binary existence/non-existence); extensions to $n = 2^k$ consistent; no core claim requires $n > 2$
Decoupling of Q_2 and H (R1)	Valid — Structural; Q_2 and H are categorically distinct; T is the bridge
Import I-A from ZP-A	Valid — Received; CC-1 reclassification noted
Import I-B from ZP-B	Valid — Received; MP-1 included; C3 noted
DP-1: Orthogonality	Valid — Design Principle; reclassified from T1; well-motivated and explicit
D2: T requirements	Valid — Defined; five requirements stated; all satisfied by T2
T2: Existence of T	Valid — Derived; basis assignment; all five requirements verified
T3: Uniqueness of T	Valid — Proposition; derived; unique up to unitary equivalence
T4: Snap → orthogonal shift	Valid — Theorem; derived; unconditional; depends on DP-1
T5: Monotone norms	Valid — Proposition; derived; unconditional; from T2 and ZP-A T3