THE ZERO PARADOX

ZP-F: The Counterexamples

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This document is self-contained within ordered field theory. No p-adic topology, no information theory, and no Hilbert space machinery is required. All results follow from the axioms of a linearly ordered field alone.

Illustrated Companion: A paired ZP-F Illustrated Companion provides plain-language explanation and motivation for the results here. Examples and intuitions are kept separate from the formal layers to distinguish illustrative material from proofs.

I. Purpose and Scope

The Zero Paradox framework requires a metric space in which a minimal first departure from the null state is structurally forced — the Binary Snap. This document establishes the negative result: no linearly ordered field can serve as that substrate. The real numbers ℝ are the canonical and most familiar instance; the theorem applies to any field carrying a compatible linear order.

The blocking mechanism is elementary: in any such field, for any positive element ε, the element ε/2 exists, is positive, and is strictly smaller. There is no floor. The Binary Snap requires a floor. Therefore the snap is structurally impossible in any linearly ordered field.

This result contextualises the choice of Q_2 in ZP-B. Among all completions of ℚ, Ostrowski's theorem identifies exactly two kinds: Archimedean completions (such as ℝ, where zero is a limit point and density excludes any floor) and non-Archimedean completions ($\mathbb{Q}_p$, where zero is topologically isolated by the p-adic valuation). Q_2 is the non-Archimedean completion at p = 2 — the minimum prime compatible with binary existence (AX-B1). The structural isolation of zero is not imposed from outside; it follows from the completion.

II. The General Result

The following results hold for any type F carrying a field structure, a linear order, and the IsStrictOrderedRing property — the standard Mathlib replacement for the deprecated LinearOrderedField typeclass (deprecated October 2025). All proofs are machine-verified in Lean 4.

Theorem F-DENSITY
$\forall\, \varepsilon : F,\ 0 < \varepsilon \rightarrow 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$
For any positive element ε in a linearly ordered field F, the element ε/2 is also positive and strictly smaller than ε.
Proof: half_pos gives 0 < ε/2; half_lt_self gives ε/2 < ε. Both follow from the ordered field axioms. Status: DERIVED. ✓

Theorem F-NO-MIN

$\neg\exists\, \varepsilon : F,\ 0 < \varepsilon \wedge \forall\, \delta : F,\ 0 < \delta \to \varepsilon \le \delta$

No linearly ordered field has a minimal positive element.

Proof: Suppose such ε exists. By F-DENSITY, $0 < \varepsilon/2 < \varepsilon$. Then $\varepsilon \le \varepsilon/2$ (by minimality) contradicts $\varepsilon/2 < \varepsilon$. Status: DERIVED. ✓

Theorem F-SNAP-BLOCKED

$\forall\, \varepsilon_0 : F,\ 0 < \varepsilon_0 \to \exists\, \delta : F,\ 0 < \delta \wedge \delta < \varepsilon_0$

Any candidate first step $\varepsilon_0 > 0$ from 0 in a linearly ordered field is blocked: $\varepsilon_0/2$ is a smaller positive element, so ε_0 cannot be a genuine first step.

Proof: Take $\delta = \varepsilon_0/2$. F-DENSITY gives $0 < \delta < \varepsilon_0$. Status: DERIVED. ✓

Theorem F-SNAP-IMPOSSIBLE

$\neg\exists\, \varepsilon_0 : F,\ 0 < \varepsilon_0 \wedge \neg\exists\, \delta : F,\ 0 < \delta \wedge \delta < \varepsilon_0$

The Binary Snap cannot occur in any linearly ordered field.

A snap requires a minimal first departure from 0 with nothing below it. F-SNAP-BLOCKED shows such a floor never exists in a linearly ordered field.

Proof: Direct from F-SNAP-BLOCKED. Status: DERIVED. ✓

III. The Real Numbers — Canonical Instance

The following are corollaries of the general results above, instantiated at $F = \mathbb{R}$. They are stated explicitly because $\mathbb{R}$ is the most familiar and intuitive case, and because the ZP-F companion document is written around $\mathbb{R}$ as its primary example.

Corollary R-DENSITY

$\forall\, \varepsilon : \mathbb{R},\ 0 < \varepsilon \to 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$

Density at zero for $\mathbb{R}$. Follows from F-DENSITY at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-NO-MIN

$\neg\exists\, \varepsilon : \mathbb{R},\ 0 < \varepsilon \wedge \forall\, \delta : \mathbb{R},\ 0 < \delta \to \varepsilon \le \delta$

$\mathbb{R}$ has no minimal positive element. Follows from F-NO-MIN at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-SNAP-BLOCKED

$\forall\, \varepsilon_0 : \mathbb{R},\ 0 < \varepsilon_0 \to \exists\, \delta : \mathbb{R},\ 0 < \delta \wedge \delta < \varepsilon_0$

Corollary R-SNAP-BLOCKED
Any candidate first step in ℝ can be halved. Follows from F-SNAP-BLOCKED at F = ℝ. Status: DERIVED (corollary). ✓

Corollary R-SNAP-IMPOSSIBLE
$\neg\exists\ \varepsilon_0 : \mathbb{R},\ 0 < \varepsilon_0 \land \neg\exists\ \delta : \mathbb{R},\ 0 < \delta \land \delta < \varepsilon_0$
The Binary Snap cannot occur in ℝ. Follows from F-SNAP-IMPOSSIBLE at F = ℝ. Status: DERIVED (corollary). ✓

IV. Axiom Profile

All eight results (F-DENSITY, F-NO-MIN, F-SNAP-BLOCKED, F-SNAP-IMPOSSIBLE and their ℝ corollaries) verify under #print axioms with the profile: propext, Classical.choice, Quot.sound. This is the standard profile for Mathlib field and order results. No snap-specific axioms are introduced.

V. Relationship to the Framework

ZP-F is self-contained and does not feed formally into any other layer. It is the negative complement to ZP-B: where ZP-B establishes that Q_2 can host the snap (via the 2-adic valuation and clopen ball structure), ZP-F establishes that ℝ — and any linearly ordered field — cannot.

The dependency order of the framework places ZP-F as independent:

Layer	Role	Dependency
ZP-B	p-Adic topology — Q_2 hosts the snap	ZP-A
ZP-F	Counterexample — ℝ and ordered fields cannot host the snap	None (self-contained)
ZP-G	Category theory	Self-contained; ZP-E conceptually

VI. Validation Status

Component	Status / Notes
F-DENSITY	DERIVED — Lean-verified; general case
F-NO-MIN	DERIVED — Lean-verified; general case
F-SNAP-BLOCKED	DERIVED — Lean-verified; general case
F-SNAP-IMPOSSIBLE	DERIVED — Lean-verified; general case
R-DENSITY	DERIVED — Corollary of F-DENSITY at F = ℝ
R-NO-MIN	DERIVED — Corollary of F-NO-MIN at F = ℝ
R-SNAP-BLOCKED	DERIVED — Corollary of F-SNAP-BLOCKED at F = ℝ

Component	Status / Notes
R-SNAP-IMPOSSIBLE	DERIVED — Corollary of F-SNAP-IMPOSSIBLE at $F = \mathbb{R}$