

# THE ZERO PARADOX

*ZP-F: The Counterexamples*

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This document is self-contained within ordered field theory. No p-adic topology, no information theory, and no Hilbert space machinery is required. All results follow from the axioms of a linearly ordered field alone.

Illustrated Companion: A paired ZP-F Illustrated Companion provides plain-language explanation and motivation for the results here. Examples and intuitions are kept separate from the formal layers to distinguish illustrative material from proofs.

## I. Purpose and Scope

The Zero Paradox framework requires a metric space in which a minimal first departure from the null state is structurally forced — the Binary Snap. This document establishes the negative result: no linearly ordered field can serve as that substrate. The real numbers  $\mathbb{R}$  are the canonical and most familiar instance; the theorem applies to any field carrying a compatible linear order.

The blocking mechanism is elementary: in any such field, for any positive element  $\varepsilon$ , the element  $\varepsilon/2$  exists, is positive, and is strictly smaller. There is no floor. The Binary Snap requires a floor. Therefore the snap is structurally impossible in any linearly ordered field.

This result contextualises the choice of  $\mathbb{Q}_2$  in ZP-B. Among all completions of  $\mathbb{Q}$ , Ostrowski's theorem identifies exactly two kinds: Archimedean completions (such as  $\mathbb{R}$ , where zero is a limit point and density excludes any floor) and non-Archimedean completions ( $\mathbb{Q}_p$ , where zero is topologically isolated by the p-adic valuation).  $\mathbb{Q}_2$  is the non-Archimedean completion at  $p = 2$  — the minimum prime compatible with binary existence (AX-B1). The structural isolation of zero is not imposed from outside; it follows from the completion.

## II. The General Result

The following results hold for any type  $F$  carrying a field structure, a linear order, and the `IsStrictOrderedRing` property — the standard Mathlib replacement for the deprecated `LinearOrderedField` typeclass (deprecated October 2025). All proofs are machine-verified in Lean 4.

### Theorem F-DENSITY

$$\forall \varepsilon : F, 0 < \varepsilon \rightarrow 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$$

For any positive element  $\varepsilon$  in a linearly ordered field  $F$ , the element  $\varepsilon/2$  is also positive and strictly smaller than  $\varepsilon$ .

Proof: `half_pos` gives  $0 < \varepsilon/2$ ; `half_lt_self` gives  $\varepsilon/2 < \varepsilon$ . Both follow from the ordered field axioms. Status: DERIVED. ✓

### Theorem F-NO-MIN

$$\neg \exists \varepsilon : F, 0 < \varepsilon \wedge \forall \delta : F, 0 < \delta \rightarrow \varepsilon \leq \delta$$

No linearly ordered field has a minimal positive element.

Proof: Suppose such  $\varepsilon$  exists. By F-DENSITY,  $0 < \varepsilon/2 < \varepsilon$ . Then  $\varepsilon \leq \varepsilon/2$  (by minimality) contradicts  $\varepsilon/2 < \varepsilon$ . Status: DERIVED. ✓

### Theorem F-SNAP-BLOCKED

$$\forall \varepsilon_0 : F, 0 < \varepsilon_0 \rightarrow \exists \delta : F, 0 < \delta \wedge \delta < \varepsilon_0$$

Any candidate first step  $\varepsilon_0 > 0$  from 0 in a linearly ordered field is blocked:  $\varepsilon_0/2$  is a smaller positive element, so  $\varepsilon_0$  cannot be a genuine first step.

Proof: Take  $\delta = \varepsilon_0/2$ . F-DENSITY gives  $0 < \delta < \varepsilon_0$ . Status: DERIVED. ✓

### Theorem F-SNAP-IMPOSSIBLE

$$\neg \exists \varepsilon_0 : F, 0 < \varepsilon_0 \wedge \neg \exists \delta : F, 0 < \delta \wedge \delta < \varepsilon_0$$

The Binary Snap cannot occur in any linearly ordered field.

A snap requires a minimal first departure from 0 with nothing below it. F-SNAP-BLOCKED shows such a floor never exists in a linearly ordered field.

Proof: Direct from F-SNAP-BLOCKED. Status: DERIVED. ✓

## III. The Real Numbers — Canonical Instance

The following are corollaries of the general results above, instantiated at  $F = \mathbb{R}$ . They are stated explicitly because  $\mathbb{R}$  is the most familiar and intuitive case, and because the ZP-F companion document is written around  $\mathbb{R}$  as its primary example.

### Corollary R-DENSITY

$$\forall \varepsilon : \mathbb{R}, 0 < \varepsilon \rightarrow 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$$

Density at zero for  $\mathbb{R}$ . Follows from F-DENSITY at  $F = \mathbb{R}$ . Status: DERIVED (corollary). ✓

### Corollary R-NO-MIN

$$\neg \exists \varepsilon : \mathbb{R}, 0 < \varepsilon \wedge \forall \delta : \mathbb{R}, 0 < \delta \rightarrow \varepsilon \leq \delta$$

$\mathbb{R}$  has no minimal positive element. Follows from F-NO-MIN at  $F = \mathbb{R}$ . Status: DERIVED (corollary). ✓

### Corollary R-SNAP-BLOCKED

$$\forall \varepsilon_0 : \mathbb{R}, 0 < \varepsilon_0 \rightarrow \exists \delta : \mathbb{R}, 0 < \delta \wedge \delta < \varepsilon_0$$

### Corollary R-SNAP-BLOCKED

Any candidate first step in  $\mathbb{R}$  can be halved. Follows from F-SNAP-BLOCKED at  $F = \mathbb{R}$ . Status: DERIVED (corollary).  
✓

### Corollary R-SNAP-IMPOSSIBLE

$\neg \exists \varepsilon_0 : \mathbb{R}, 0 < \varepsilon_0 \wedge \neg \exists \delta : \mathbb{R}, 0 < \delta \wedge \delta < \varepsilon_0$

The Binary Snap cannot occur in  $\mathbb{R}$ . Follows from F-SNAP-IMPOSSIBLE at  $F = \mathbb{R}$ . Status: DERIVED (corollary). ✓

## IV. Axiom Profile

All eight results (F-DENSITY, F-NO-MIN, F-SNAP-BLOCKED, F-SNAP-IMPOSSIBLE and their  $\mathbb{R}$  corollaries) verify under #print axioms with the profile: propext, Classical.choice, Quot.sound. This is the standard profile for Mathlib field and order results. No snap-specific axioms are introduced.

## V. Relationship to the Framework

ZP-F is self-contained and does not feed formally into any other layer. It is the negative complement to ZP-B: where ZP-B establishes that  $\mathbb{Q}_2$  can host the snap (via the 2-adic valuation and clopen ball structure), ZP-F establishes that  $\mathbb{R}$  — and any linearly ordered field — cannot.

The dependency order of the framework places ZP-F as independent:

Layer	Role	Dependency
ZP-B	p-Adic topology — $\mathbb{Q}_2$ hosts the snap	ZP-A
ZP-F	Counterexample — $\mathbb{R}$ and ordered fields cannot host the snap	None (self-contained)
ZP-G	Category theory	Self-contained; ZP-E conceptually

## VI. A Philosophical Note: The Wrong Setting

The real numbers are the most natural and familiar number line precisely because they fill every gap. Density — the property established in F-DENSITY: for any  $\varepsilon > 0$  there exists  $\delta$  with  $0 < \delta < \varepsilon$  — is not a deficiency in  $\mathbb{R}$ ; it is one of  $\mathbb{R}$ 's defining virtues. The counterexample above shows that this virtue is exactly what disqualifies  $\mathbb{R}$  as a host for the Binary Snap. The exclusion is not a matter of degree: density follows from the ordered field axioms, so every linearly ordered field is ruled out by exactly the same argument. Familiarity here misleads: the setting that feels most natural for foundational mathematics is precisely the wrong one.

The exclusion of  $\mathbb{R}$  is not an isolated fact about one number system. F-SNAP-IMPOSSIBLE applies to any linearly ordered field. By Ostrowski's theorem, every completion of  $\mathbb{Q}$  falls into exactly one of two classes: Archimedean (including  $\mathbb{R}$ ) or non-Archimedean (the p-adic fields  $\mathbb{Q}_p$ ). The theorems of this document eliminate the entire Archimedean class. What remains is the non-Archimedean class. The framework's binary existence constraint (AX-B1) then selects  $p = 2$  as the minimum prime.  $\mathbb{Q}_2$  is not chosen against  $\mathbb{R}$ ; it

is what remains after  $\mathbb{R}$  — and every Archimedean completion — has been ruled out.

The structural difference that the Ostrowski classification makes precise can be stated directly. In any Archimedean completion, zero is a limit point: every neighbourhood of zero contains a smaller positive element. The snap requires zero to be something different — a structural origin from which the first departure is forced, with no smaller departure possible. In  $\mathbb{Q}_2$ , zero carries the infinite 2-adic valuation; it is valuatively distinguished in exactly the sense required. The snap does not fail in  $\mathbb{R}$  by accident. It fails because  $\mathbb{R}$  and the snap require incompatible roles for zero: limit point versus structural origin.

### Remark: The Dual Limit Condition

The results of §II–§III and ZP-L together admit an informal structural account of why the snap is domain-specific.

In any dense ordered field, 0 has no nearest neighbour from above: for any  $\varepsilon > 0$  there exists  $\delta$  with  $0 < \delta < \varepsilon$  (F-DENSITY). Zero is a topological limit point from above with no minimum positive element. The snap requires a discrete first step above 0; density forbids it. This is the blocking condition established in this document.

$\varepsilon_0$  (the snap threshold, established in ZP-L) is the least fixed point of  $\alpha \mapsto \omega^\alpha$ : it is not reachable by any finite iteration of the  $\omega^\alpha(\cdot)$  operation, only by the limit of the tower  $\omega, \omega^\omega, \omega^{\omega^\omega}, \dots$ . The snap is possible at  $\varepsilon_0$  precisely because it has no predecessor in the ordinal hierarchy: no ordinal  $\alpha$  satisfies  $\alpha + 1 = \varepsilon_0$ .

Both cases are limit-type conditions: the field case is a topological limit point with no minimum positive neighbour above 0; the ordinal case is a limit ordinal (specifically, the least fixed point of  $\alpha \mapsto \omega^\alpha$ ) with no predecessor below  $\varepsilon_0$ . The analogy is structural — both describe a boundary approached but not reachable by finite steps on one side — but it is not a proved identification. The two structures live in different mathematical categories.

A domain that can host the snap must satisfy two conditions simultaneously: (1) the boundary is a genuine limit (no nearest neighbour — what makes the threshold non-arbitrary) and (2) the crossing is discrete (a step, not a continuous transition). Dense ordered fields have condition (1) but density makes condition (2) impossible. In the non-standard naturals  ${}^*\mathbb{N}$  with the order topology, every element has an immediate predecessor, so there is no limit point at infinity in the order-topology sense; condition (1) fails informally. (This  ${}^*\mathbb{N}$  observation is informal — not a proved result of this document or ZP-L.)

The tower encodings  $\text{cnfToZp2}(\text{towerNONote } n)$  converge to 0 in  $\mathbb{Z}_2$  as  $n \rightarrow \infty$  (proved in ZP-L), and  $\varepsilon_0$  is the minimal snap threshold in the ordinal setting (proved in ZP-L). The structural identification of these two results — that  $\mathbb{Z}_2$ 's limit at 0 and the ordinal threshold at  $\varepsilon_0$  reflect the same boundary — has a remaining gap: no type bridge between the ordinal and p-adic types is established in ZP-L (see ZP-L §V, Remaining Gap).

This is an observation, not a proved theorem. The blocking result (F-SNAP-IMPOSSIBLE) is proved here; the threshold and convergence results are proved in ZP-L. The claim that both conditions reflect a common structural property is interpretive — it does not follow from their conjunction alone. The  ${}^*\mathbb{N}$  example above is informal.

The dual-approach structure recurs across the ZP layers: in ZP-L,  $\varepsilon_0$  is located from above as the least fixed point of  $\alpha \mapsto \omega^\alpha$  and from below as the limit of the CNF tower; in ZP-M, the computability and ordinal fixed points are argued to converge to the same 2-adic limit. This recurrence is not coincidental. The snap is the point where a domain's own measurement framework exhausts itself; no single framework can locate it from inside because the framework runs out exactly there. This suggests a dual approach is required in each case: each framework reaches the boundary from its own direction, and the snap is the point they agree on. No general theorem to this effect is established here.

### Remark: The Dual Limit Condition

The surreal numbers (No) satisfy both conditions — non-Archimedean structure (containing  $\varepsilon_0$  as an ordinal) and genuine limit ordinals — and are the natural boundary test case for any duality theorem. Whether the surreals admit a snap-like result, or whether a third structural condition is required to exclude them, has not been determined.

(Informal — not a result of this document.)

Development path toward a formal theorem: prove the two conditions as independent Lean results (density impossibility established here; limit ordinal necessity established in ZP-L), then prove they agree on the same structural point. The surreal case is the critical test of whether two conditions are jointly sufficient.

## VII. Validation Status

Component	Status / Notes
F-DENSITY	DERIVED — Lean-verified; general case
F-NO-MIN	DERIVED — Lean-verified; general case
F-SNAP-BLOCKED	DERIVED — Lean-verified; general case
F-SNAP-IMPOSSIBLE	DERIVED — Lean-verified; general case
R-DENSITY	DERIVED — Corollary of F-DENSITY at $F = \mathbb{R}$
R-NO-MIN	DERIVED — Corollary of F-NO-MIN at $F = \mathbb{R}$
R-SNAP-BLOCKED	DERIVED — Corollary of F-SNAP-BLOCKED at $F = \mathbb{R}$
R-SNAP-IMPOSSIBLE	DERIVED — Corollary of F-SNAP-IMPOSSIBLE at $F = \mathbb{R}$