THE ZERO PARADOX

ZP-F: The Counterexamples

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This document is self-contained within ordered field theory. No p-adic topology, no information theory, and no Hilbert space machinery is required. All results follow from the axioms of a linearly ordered field alone.

Illustrated Companion: A paired ZP-F Illustrated Companion provides plain-language explanation and motivation for the results here. Examples and intuitions are kept separate from the formal layers to distinguish illustrative material from proofs.

I. Purpose and Scope

The Zero Paradox framework requires a metric space in which a minimal first departure from the null state is structurally forced — the Binary Snap. This document establishes the negative result: no linearly ordered field can serve as that substrate. The real numbers ℝ are the canonical and most familiar instance; the theorem applies to any field carrying a compatible linear order.

The blocking mechanism is elementary: in any such field, for any positive element ε, the element ε/2 exists, is positive, and is strictly smaller. There is no floor. The Binary Snap requires a floor. Therefore the snap is structurally impossible in any linearly ordered field.

This result contextualises the choice of Q_2 in ZP-B. Among all completions of ℚ, Ostrowski's theorem identifies exactly two kinds: Archimedean completions (such as ℝ, where zero is a limit point and density excludes any floor) and non-Archimedean completions ($\mathbb{Q}_p$, where zero is valuatively distinguished ($v_p(0) = +\infty$) from all nonzero elements). Q_2 is the non-Archimedean completion at p = 2 — the minimum prime compatible with binary existence (AX-B1). The valuative gap at zero is not imposed from outside; it follows from the completion.

II. The General Result

The following results hold for any type F carrying a field structure, a linear order, and the IsStrictOrderedRing property — the standard Mathlib replacement for the deprecated LinearOrderedField typeclass (deprecated October 2025). All proofs are machine-verified in Lean 4.

Theorem F-DENSITY
$\forall \varepsilon : F,\ 0 < \varepsilon \to 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$
For any positive element ε in a linearly ordered field F, the element ε/2 is also positive and strictly smaller than ε.
Proof: half_pos gives 0 < ε/2; half_lt_self gives ε/2 < ε. Both follow from the ordered field axioms. Status: DERIVED. ✓

Theorem F-NO-MIN

$\neg\exists\, \varepsilon : F, 0 < \varepsilon \wedge \forall\, \delta : F, 0 < \delta \to \varepsilon \leq \delta$

No linearly ordered field has a minimal positive element.

Proof: Suppose such ε exists. By F-DENSITY, $0 < \varepsilon/2 < \varepsilon$. Then $\varepsilon \leq \varepsilon/2$ (by minimality) contradicts $\varepsilon/2 < \varepsilon$. Status: DERIVED. ✓

Theorem F-SNAP-BLOCKED

$\forall\, \varepsilon_0 : F, 0 < \varepsilon_0 \to \exists\, \delta : F, 0 < \delta \wedge \delta < \varepsilon_0$

Any candidate first step $\varepsilon_0 > 0$ from 0 in a linearly ordered field is blocked: $\varepsilon_0/2$ is a smaller positive element, so ε_0 cannot be a genuine first step.

Proof: Take $\delta = \varepsilon_0/2$. F-DENSITY gives $0 < \delta < \varepsilon_0$. Status: DERIVED. ✓

Theorem F-SNAP-IMPOSSIBLE

$\neg\exists\, \varepsilon_0 : F, 0 < \varepsilon_0 \wedge \neg\exists\, \delta : F, 0 < \delta \wedge \delta < \varepsilon_0$

The Binary Snap cannot occur in any linearly ordered field.

A snap requires a minimal first departure from 0 with nothing below it. F-SNAP-BLOCKED shows such a floor never exists in a linearly ordered field.

Proof: Direct from F-SNAP-BLOCKED. Status: DERIVED. ✓

III. The Real Numbers — Canonical Instance

The following are corollaries of the general results above, instantiated at $F = \mathbb{R}$. They are stated explicitly because ℝ is the most familiar and intuitive case, and because the ZP-F companion document is written around ℝ as its primary example.

Corollary R-DENSITY

$\forall\, \varepsilon : \mathbb{R}, 0 < \varepsilon \to 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$

Density at zero for ℝ. Follows from F-DENSITY at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-NO-MIN

$\neg\exists\, \varepsilon : \mathbb{R}, 0 < \varepsilon \wedge \forall\, \delta : \mathbb{R}, 0 < \delta \to \varepsilon \leq \delta$

ℝ has no minimal positive element. Follows from F-NO-MIN at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-SNAP-BLOCKED

$\forall\, \varepsilon_0 : \mathbb{R}, 0 < \varepsilon_0 \to \exists\, \delta : \mathbb{R}, 0 < \delta \wedge \delta < \varepsilon_0$

Corollary R-SNAP-BLOCKED

Any candidate first step in ℝ can be halved. Follows from F-SNAP-BLOCKED at F = ℝ. Status: DERIVED (corollary). ✓

Corollary R-SNAP-IMPOSSIBLE

$\neg\exists\, \varepsilon_0 : \mathbb{R},\ 0 < \varepsilon_0 \wedge \neg\exists\, \delta : \mathbb{R},\ 0 < \delta \wedge \delta < \varepsilon_0$

The Binary Snap cannot occur in ℝ. Follows from F-SNAP-IMPOSSIBLE at F = ℝ. Status: DERIVED (corollary). ✓

IV. Axiom Profile

All eight results (F-DENSITY, F-NO-MIN, F-SNAP-BLOCKED, F-SNAP-IMPOSSIBLE and their ℝ corollaries) verify under #print axioms with the profile: propext, Classical.choice, Quot.sound. This is the standard profile for Mathlib field and order results. No snap-specific axioms are introduced.

V. Relationship to the Framework

ZP-F is self-contained and does not feed formally into any other layer. It is the negative complement to ZP-B: where ZP-B establishes that Q_2 can host the snap (via the 2-adic valuation and clopen ball structure), ZP-F establishes that ℝ — and any linearly ordered field — cannot.

The dependency order of the framework places ZP-F as independent:

Layer	Role	Dependency
ZP-B	p-Adic topology — Q_2 hosts the snap	ZP-A
ZP-F	Counterexample — ℝ and ordered fields cannot host the snap	None (self-contained)
ZP-G	Category theory	Self-contained; ZP-E conceptually

VI. A Philosophical Note: The Wrong Setting

The real numbers are the most natural and familiar number line precisely because they fill every gap. Density — the property established in F-DENSITY: for any $\varepsilon > 0$ there exists δ with $0 < \delta < \varepsilon$ — is not a deficiency in ℝ; it is one of ℝ's defining virtues. The counterexample above shows that this virtue is exactly what disqualifies ℝ as a host for the Binary Snap. The exclusion is not a matter of degree: density follows from the ordered field axioms, so every linearly ordered field is ruled out by exactly the same argument. Familiarity here misleads: the setting that feels most natural for foundational mathematics is precisely the wrong one.

The exclusion of ℝ is not an isolated fact about one number system. F-SNAP-IMPOSSIBLE applies to any linearly ordered field. By Ostrowski's theorem, every completion of ℚ falls into exactly one of two classes: Archimedean (including ℝ) or non-Archimedean (the p-adic fields ℚ[?]). The theorems of this document eliminate the entire Archimedean class. What remains is the non-Archimedean class. The framework's binary existence constraint (AX-B1) then selects $p = 2$ as the minimum prime. Q_2 is not chosen against ℝ; it is what remains after ℝ — and every Archimedean completion — has been ruled out.

The structural difference that the Ostrowski classification makes precise can be stated directly. In any Archimedean completion, zero is a limit point: every neighbourhood of zero contains a smaller positive element. The snap requires zero to be something different — a structural origin from which the first departure is forced, with no smaller departure possible. In Q_2, zero carries the infinite 2-adic valuation; it is valuatively distinguished in exactly the sense required. The snap does not fail in ℝ by accident. It fails because ℝ and the snap require incompatible roles for zero: limit point versus structural origin.

Remark: The Dual Limit Condition

The results of §II–§III and ZP-L together admit an informal structural account of why the snap is domain-specific.

In any dense ordered field, 0 has no nearest neighbour from above: for any $\varepsilon > 0$ there exists δ with $0 < \delta < \varepsilon$ (F-DENSITY). Zero is a topological limit point from above with no minimum positive element. The snap requires a discrete first step above 0; density forbids it. This is the blocking condition established in this document.

ε_0 (the snap threshold, established in ZP-L) is the least fixed point of α ↦ ω^α: it is not reachable by any finite iteration of the ω^(·) operation, only by the limit of the tower ω, ω^ω, ω^ω^ω, … The snap is possible at ε_0 precisely because it has no predecessor in the ordinal hierarchy: no ordinal α satisfies $\alpha + 1 = \varepsilon_0$.

Both cases are limit-type conditions: the field case is a topological limit point with no minimum positive neighbour above 0; the ordinal case is a limit ordinal (specifically, the least fixed point of α ↦ ω^α) with no predecessor below ε_0. The analogy is structural — both describe a boundary approached but not reachable by finite steps on one side — but it is not a proved identification. The two structures live in different mathematical categories.

A domain that can host the snap must satisfy two conditions simultaneously: (1) the boundary is a genuine limit (no nearest neighbour — what makes the threshold non-arbitrary) and (2) the crossing is discrete (a step, not a continuous transition). Dense ordered fields have condition (1) but density makes condition (2) impossible. In the non-standard naturals *ℕ with the order topology, every element has an immediate predecessor, so there is no limit point at infinity in the order-topology sense; condition (1) fails informally. (This *ℕ observation is informal — not a proved result of this document or ZP-L.)

The tower encodings cnfToZp2(towerNONote n) converge to 0 in $\mathbb{Z}_2$ as $n \to \infty$ (proved in ZP-L), and ε_0 is the minimal snap threshold in the ordinal setting (proved in ZP-L). The structural identification of these two results — that $\mathbb{Z}_2$'s limit at 0 and the ordinal threshold at ε_0 reflect the same boundary — has a remaining gap: no type bridge between the ordinal and p-adic types is established in ZP-L (see ZP-L §V, Remaining Gap).

This is an observation, not a proved theorem. The blocking result (F-SNAP-IMPOSSIBLE) is proved here; the threshold and convergence results are proved in ZP-L. The claim that both conditions reflect a common structural property is interpretive — it does not follow from their conjunction alone. The *ℕ example above is informal.

The dual-approach structure recurs across the ZP layers: in ZP-L, ε_0 is located from above as the least fixed point of α ↦ ω^α and from below as the limit of the CNF tower; in ZP-M, the computability and ordinal fixed points are argued to converge to the same 2-adic limit. This recurrence is not coincidental. The snap is the point where a domain's own measurement framework exhausts itself; no single framework can locate it from inside because the framework runs out exactly there. This suggests a dual approach is required in each case: each framework reaches the boundary from its own direction, and the snap is the point they agree on. No general theorem to this effect is established here.

Remark: The Dual Limit Condition

The surreal numbers (No) satisfy both conditions — non-Archimedean structure (containing ε_0 as an ordinal) and genuine limit ordinals — and are the natural boundary test case for any duality theorem. Whether the surreals admit a snap-like result, or whether a third structural condition is required to exclude them, has not been determined. (Informal — not a result of this document.)

Development path toward a formal theorem: prove the two conditions as independent Lean results (density impossibility established here; limit ordinal necessity established in ZP-L), then prove they agree on the same structural point. The surreal case is the critical test of whether two conditions are jointly sufficient.

VII. Validation Status

Component	Status / Notes
F-DENSITY	DERIVED — Lean-verified; general case
F-NO-MIN	DERIVED — Lean-verified; general case
F-SNAP-BLOCKED	DERIVED — Lean-verified; general case
F-SNAP-IMPOSSIBLE	DERIVED — Lean-verified; general case
R-DENSITY	DERIVED — Corollary of F-DENSITY at $F = \mathbb{R}$
R-NO-MIN	DERIVED — Corollary of F-NO-MIN at $F = \mathbb{R}$
R-SNAP-BLOCKED	DERIVED — Corollary of F-SNAP-BLOCKED at $F = \mathbb{R}$
R-SNAP-IMPOSSIBLE	DERIVED — Corollary of F-SNAP-IMPOSSIBLE at $F = \mathbb{R}$