# THE ZERO PARADOX

## ZP-G: Category Theory

*Version 1.1 | April 2026*

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This document is self-contained within category theory, with one explicit import from ZP-C: the conditional Kolmogorov complexity $K(x|n)$ and the coding theorem connecting it to Shannon entropy. That import is named and labelled; it replaces the v1.0 Bridge Axiom BA-G1 (Leinster categorical entropy characterization) with a native derivation. All other content from ZP-A, ZP-B, ZP-D, and ZP-E remains excluded from this document. Cross-framework connections are deferred to ZP-H.

Honest labelling is the governing discipline. Every claim is marked as Axiom, Definition, Derived, Import, Design Commitment, or Remark. Nothing slides between categories.

Version 1.1 changes from v1.0:

*(1) D7 (categorical surprisal via Shannon entropy functor H) is replaced by D7' (conditional Kolmogorov complexity $K(B|A)$ as native categorical surprisal). D7' requires no external characterization of entropy.*

*(2) Import I-KC added: $K(x|n)$ and the Kolmogorov coding theorem are imported from ZP-C as a named dependency. ZP-G is no longer purely categorical; this dependency is explicitly stated rather than concealed.*

*(3) BA-G1 (Leinster Bridge Axiom) is demoted to Remark R-BA: a compatibility result showing that D7' and the Shannon functor are equivalent under the coding theorem. BA-G1 is no longer a premise of any theorem.*

*(4) T6 (informational singularity) is rebuilt on D7'. The proof is now self-contained within ZP-G plus the named ZP-C import. OQ-G1 is closed.*

*(5) All theorems from v1.0 that did not depend on BA-G1 (T1 through T5, T7) are unchanged in statement and proof. Their validation status is carried forward.*

## I. Categorical Primitives

Definitions D1 through D6 and Theorems T1 through T5 are unchanged from v1.0. They are reproduced here for completeness and self-reference.

### 1.1 The Definition of a Category

| **Definition D1 — Category** |
|---|
| *Status: Definition — foundational* |
| A category C consists of: |
| (i) Objects: A collection ob(C), written A, B, X, 0, … |
| (ii) Morphisms: For each ordered pair (A, B), a collection hom(A, B) of morphisms, written $f: A \to B$. |
| (iii) Composition: $\circ$: hom(B,C) × hom(A,B) → hom(A,C), written $g \circ f$. |
| (iv) Identity: For each A, a morphism $id_A: A \to A$. |
| Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$. |
| Unit laws: $id_B \circ f = f = f \circ id_A$. |

| **Definition D2 — Morphism Uniqueness Notation** |
|---|
| *Status: Definition* |
| A morphism $f: A \to B$ is unique if for any $g, h: A \to B$, $g = h$. Written: $\exists! f: A \to B$. |

### 1.2 Initial and Terminal Objects

| **Definition D3 — Initial Object** |
|---|
| *Status: Definition — load-bearing* |
| An object $0 \in ob(C)$ is initial if for every $X \in ob(C)$, there exists a unique morphism $\iota_X: 0 \to X$. |

| **Theorem T1 — Uniqueness of the Initial Object** |
|---|
| *Status: Derived — standard category theory [unchanged from v1.0]* |
| If 0 and 0' are both initial in C, ∃! isomorphism $0 \cong 0'$. The initial object is unique up to unique isomorphism. |
| Proof: $\exists! f: 0 \to 0'$ and $\exists! g: 0' \to 0$. Initiality forces $g \circ f = id_0$ and $f \circ g = id_{0'}$. Therefore f is a unique isomorphism. ✓ |

**Definition D4 — Terminal Object**

*Status: Definition — defined for exclusion*

An object $1 \in \mathrm{ob}(C)$ is terminal if for every $X \in \mathrm{ob}(C)$, $\exists!$ morphism $\tau_X: X \to 1$. The Zero Paradox framework is not a zero object category ($0 \not\cong 1$).

## II. The Foundational Axioms of ZP-G

| **Axiom AX-G1 — Asymmetry Axiom** |
|---|
| *Status: Axiom — foundational structural commitment [unchanged from v1.0]* |
| The category C possesses an initial object 0 and no terminal object. |
| Formally: ∃ 0 ∈ ob(C) satisfying D3. ¬∃ 1 ∈ ob(C) satisfying D4. |
| Correspondence: In ZP-A: join-semilattice without ⊤ and without ∧. In ZP-B: $Q_2$ has no element to which all paths converge. The present axiom is the categorical generalization. |

| **Axiom AX-G2 — Source Asymmetry** |
|---|
| *Status: Axiom — foundational [unchanged from v1.0]* |
| For any non-initial object X ≇ 0: hom(X, 0) = ∅. |
| Motivation: The categorical expression of irreversibility. Morphisms $\iota_X$: 0 → X exist for all X. Their reversal does not exist. AX-G2 is consistent with AX-G1 but not derivable from it. |

## III. Universal Constituent and Unreachability

**Theorem T2 — Universal Constituent**

*Status: Derived — from D3 and AX-G1 [unchanged from v1.0]*

For every $X \in ob(C)$, $\exists! \iota_X: 0 \to X$. The initial object 0 is the universal categorical source.

Proof: Immediate from D3 and AX-G1. ✓

**Theorem T3 — Unreachability of 0**

*Status: Derived — from AX-G2 [unchanged from v1.0]*

For any $X \not\cong 0$: $hom(X, 0) = \emptyset$. The initial object 0 is unreachable from any non-initial object.

Proof: Direct from AX-G2. ✓

**Remark R1 — Structural Inversion — The Categorical Zero Paradox**

*Status: Remark [unchanged from v1.0]*

T2 and T3 together constitute the categorical Zero Paradox. 0 reaches every object (T2); no non-initial object reaches 0 (T3). This is not a logical contradiction. It is a structural inversion: the unique universal source is the unique object with no incoming non-trivial morphisms.

## IV. Monotone Structure and the Additive Ontology

**Definition D5 — Morphism Chain**

*Status: Definition [unchanged from v1.0]*

A morphism chain of length n from 0 is a sequence:

$$0 = X_0 \to X_1 \to \dots \to X_n$$

where each $X_k \to X_{k+1}$ is a morphism in C.

**Theorem T4 — Chains are Forward-Only**

*Status: Derived — from AX-G2 [unchanged from v1.0]*

No morphism chain from 0 can return to 0 through non-initial objects.

Proof: A return morphism $X_n \to 0$ for $X_n \ncong 0$ would contradict AX-G2. ✓

**Definition D6 — Functor**

*Status: Definition — standard category theory [unchanged from v1.0]*

A functor F: C → D consists of an object map F: ob(C) → ob(D) and a morphism map F: $hom_C(A,B) \to hom_D(F(A), F(B))$, preserving composition and identity.

**Theorem T5 — Functors Preserve Initial Objects**

*Status: Derived — [OQ-G2 closed in ZP-H T-H1; unchanged from v1.0]*

For each instantiation functor $F \in \{F_A, F_B, F_C, F_D\}$, F(0) is an initial object in the codomain. Verified by direct universal property check in ZP-H T-H1. ✓

## V. The Kolmogorov Import from ZP-C

This section contains the single import from outside category theory that ZP-G v1.1 requires. It is named, scoped, and labelled explicitly. It replaces BA-G1 (the Leinster Bridge Axiom from v1.0) as the informational foundation of D7'.

| **Import I-KC — Conditional Kolmogorov Complexity from ZP-C** |
| --- |
| *Status: Import — from ZP-C D1 and standard algorithmic information theory* |
| What is imported: The conditional Kolmogorov complexity K(x\|y), defined as the length of the shortest program p such that U(p, y) = x, where U is a fixed universal Turing machine: |
| $K(x\|y) = \min \{ \|p\| : U(p, y) = x \}$ |
| Key property — The Coding Theorem: For any computable probability measure P, Kolmogorov complexity and Shannon entropy are related up to an additive constant c (depending only on U, not on x or y): |
| $K(x\|y) \approx -\log_2 P(x\|y) + O(c)$ |
| The coding theorem is a standard result of algorithmic information theory (Li and Vitányi, An Introduction to Kolmogorov Complexity and Its Applications). It is not derived within ZP-G. It is imported as a named result. |
| Scope of import: I-KC imports K(x\|y) and the coding theorem only. No other content from ZP-C is imported into ZP-G. The Kolmogorov complexity machinery is already present in ZP-C D1 (incompressibility threshold $P_o$), so I-KC introduces no new external dependency into the overall Zero Paradox framework — it only introduces a dependency within ZP-G specifically. |
| Computability note: K(x\|y) is not computable in general (it is approximable from above by standard results). This is not a defect for the present purposes: the framework requires that K(x\|y) be well-defined, not that it be computable. The ontological claims of ZP-G do not depend on computability. |
| Status: This is an import, not a bridge axiom. A bridge axiom is a claim that cannot be derived from either side and must be assumed. K(x\|y) is a defined mathematical object with a complete internal theory. I-KC is a decision to use that object within ZP-G, not an assumption about it. |

## VI. Categorical Information Theory [Rebuilt in v1.1]

### 6.1 Native Categorical Surprisal — Definition D7'

Version 1.0 defined categorical surprisal via the Shannon entropy functor H imported through BA-G1. Version 1.1 replaces this with conditional Kolmogorov complexity, imported via I-KC. The definition is native to the morphism structure of C.

| **Definition D7' — Native Categorical Surprisal** |
|---|
| *Status: Definition — from D5, I-KC [replaces D7 from v1.0]* |
| Let f: A → B be a morphism in C. Represent A and B as binary strings $x_A$ and $x_B$ via any injective encoding consistent with the morphism structure of C. The categorical surprisal of f is: |
| $I(f) = K(x_B \mid x_A)$ |
| the conditional Kolmogorov complexity of the target given the source. |
| Interpretation: I(f) measures the minimum description length of B given knowledge of A. It is the irreducible informational content added by the transition f: A → B, independent of any probability distribution. |
| Well-definedness: $K(x_B\|x_A)$ depends on the encoding of objects as strings. Different encodings yield values differing by at most an additive constant c (the coding theorem constant of I-KC). All structural claims below are invariant under this additive constant: they concern whether I(f) is finite, zero, or undefined — not its precise numerical value. |
| Relationship to v1.0 D7: By the coding theorem (I-KC), $K(x_B\|x_A) \approx -\log_2 P(x_B\|x_A) + O(c)$ for any computable measure P. Therefore D7' and D7 are equivalent up to O(c). The choice of D7' over D7 is not a change in what is being measured; it is a change in how the measure is defined — natively versus via import. |

### 6.2 Properties of the Native Surprisal

| **Theorem T6-a — Surprisal of the Identity Morphism is Zero** |
|---|
| *Status: Derived — from D7' and I-KC* |
| Claim: $I(id_A) = K(x_A\|x_A) = 0$ up to the additive constant c. |
| Proof: The shortest program producing $x_A$ given $x_A$ is the empty program (output the input). Therefore $K(x_A\|x_A) = 0$ up to c. The identity morphism adds no informational content. ✓ |

**Theorem T6-b — Surprisal is Non-Negative for Forward Morphisms**

*Status: Derived — from D5, D7', AX-G2*

Claim: For any morphism $f: A \to B$ in a forward morphism chain from 0, $I(f) \geq 0$ up to c, with strict inequality when $A \not\cong B$.

Proof: $K(x_B|x_A) \geq 0$ by definition (program length is non-negative). Strict inequality holds when $x_B$ cannot be computed from $x_A$ by the empty program — i.e., when A and B are distinct objects encoding distinct states. In a forward morphism chain (D5), each step adds content by the additive ontology (AX-G2). ✓

**Theorem T6-c — Surprisal Accumulates Along Chains**

*Status: Derived — from D5, T6-b, subadditivity of K*

Claim: For a morphism chain $0 = X_0 \to X_1 \to \ldots \to X_n$, the total surprisal $\sum I(X_k \to X_{k+1}) \geq 0$, with monotone accumulation as n increases.

Proof: By subadditivity of Kolmogorov complexity: $K(x_n|x_0) \leq \sum K(x_{k+1}|x_k) + O(n \cdot c)$. Each term is $\geq 0$ by T6-b. Adding distinct objects strictly increases the total. ✓

## VII. The Informational Singularity of 0 [Rebuilt in v1.1]

This is the central theorem of the information-theoretic section. In v1.0, it depended on BA-G1. In v1.1, it is proved from D7' and AX-G2 alone, with I-KC as the only external dependency.

| **Theorem T6 — Informational Singularity of 0** |
|---|
| *Status: Derived — from AX-G2, D7', I-KC [rebuilt from v1.0, OQ-G1 closed]* |
| Setup: Let 0 be the initial object of C (AX-G1). Let $I(f) = K(x_B \| x_A)$ be the categorical surprisal (D7'). Let I-KC provide the Kolmogorov framework. |
| Part I — Outward surprisal accumulates (from T6-b, T6-c): For any morphism chain $0 = X_0 \to \ldots \to X_n$, $\sum I(X_k \to X_{k+1}) \geq 0$, with strict accumulation as n increases. ✓ |
| Part II — Inward surprisal is undefined (from AX-G2): For any $X \not\cong 0$, $\mathrm{hom}(X, 0) = \varnothing$ (AX-G2). Therefore D7' cannot be applied to any morphism $f: X \to 0$ from outside 0 — no such morphism exists. $I(X \to 0)$ is undefined not because K diverges to infinity, but because there is no morphism to apply D7' to. The undefined-domain condition is strictly stronger than divergence. ✓ |
| Part III — The singularity: 0 is the unique object in C for which outward surprisal is defined and accumulates (Part I) while inward surprisal is undefined by absence of morphisms (Part II). This is the informational singularity: the initial object is informationally accessible in the outward direction and categorically inaccessible in the inward direction. The singularity is structural, not numerical. It does not require K to diverge — it requires only AX-G2 and D7'. ✓ |
| Comparison with ZP-C (to be reconciled in ZP-H T-H2): ZP-C establishes that the discrete surprisal DF diverges (numerically, to ∞) along infinite sequences approaching 0 in $Q_2$. T6 Part II establishes that $I(X \to 0)$ is undefined (domain-absent) for any $X \not\cong 0$ in C. These are compatible: undefined is stronger than infinite. ZP-H T-H2 proves they describe the same obstruction under the functor $F_C$. |
| *Status: DERIVED. Depends on AX-G1, AX-G2, D3, D5, D7', I-KC, T6-a, T6-b, T6-c. OQ-G1 is closed. BA-G1 is no longer a premise of T6. ✓* |

### 7.1 Compatibility with Shannon Entropy — BA-G1 Demoted to Remark

### Remark R-BA — Compatibility of D7' with the Shannon Entropy Functor

*Status: Remark — BA-G1 demoted from Bridge Axiom [v1.0] to Compatibility Remark [v1.1]*

Version 1.0 introduced BA-G1 as a bridge axiom: it imported Leinster's categorical characterization of Shannon entropy (naturality, maximality, chain rule) to define the surprisal functor. BA-G1 was the only bridge axiom in ZP-G v1.0 and was the source of OQ-G1.

In v1.1, BA-G1 is no longer a premise of any theorem. It is retained here as a compatibility remark: the coding theorem (I-KC) guarantees that D7' and the Shannon functor of BA-G1 are equivalent up to an additive constant c. Specifically:

$$K(x_B|x_A) \approx H(F(B)) - H(F(A)) + O(c)$$

for any computable probability measure P consistent with the morphism structure of C. This means all quantitative results that v1.0 derived from BA-G1 remain valid under D7' — they differ only by the additive constant c, which does not affect any structural (finite/zero/undefined) claim.

BA-G1 is not false. It is not retired. It is now a derived compatibility result rather than an assumed premise. Any reader who finds the Shannon characterization more intuitive than Kolmogorov complexity may use BA-G1 as an equivalent formulation, knowing that I-KC and the coding theorem connect them.

## VIII. The Categorical Zero Paradox — Formal Statement

Theorem T7 is the closing theorem of ZP-G. Its statement is unchanged from v1.0. Its proof is strengthened: Part IV (informational singularity) now rests on T6 as rebuilt in v1.1, which does not depend on BA-G1.

| **Theorem T7 — The Categorical Zero Paradox** |
|---|
| *Status: Derived — Closing Theorem [Part IV strengthened in v1.1]* |
| Setup: Let C satisfy AX-G1 and AX-G2. Let I be the categorical surprisal from D7'. Let I-KC provide the Kolmogorov framework. |
| Part I — Universal Constituent (T2): $\forall X \in ob(C), \exists! \iota_X : 0 \to X$. The initial object 0 is the universal categorical source. |
| Part II — Unreachability (T3): $\forall X \ncong 0$, $hom(X, 0) = \varnothing$. No non-initial object reaches 0. |
| Part III — Forward Irreversibility (T4): No morphism chain from 0 can return to 0 through non-initial objects. |
| Part IV — Informational Singularity (T6, rebuilt): $I(X \to 0)$ is undefined for all $X \ncong 0$ (no such morphism exists, AX-G2). Outward surprisal from 0 accumulates along any morphism chain (T6-b, T6-c). 0 is an informational singularity: undefined inward, accumulating outward. This part no longer depends on BA-G1. |
| Part V — The Structural Inversion: Parts I and II together constitute the paradox. 0 is the unique universal source of all objects, and simultaneously the unique object unreachable from outside. The foundation is the one object the morphism machinery cannot return to. |
| Part VI — Resolution: The paradox is not a logical contradiction. It is a structural inversion. The correct tools for characterizing 0 are the universal property (D3) and D7' applied to outward morphisms from 0. Under these tools, 0 is fully characterized. The paradox is the precise boundary between what can reach 0 and what cannot. |
| *Status: DERIVED — Closing Theorem. Depends on D3, D5, D7', AX-G1, AX-G2, I-KC, T2, T3, T4, T6. BA-G1 is not a dependency. ✓* |

## IX. Open Items Register for ZP-G v1.1

| Item | Status | Description |
|---|---|---|
| **OQ-G1** | *Closed — D7', T6* | Native categorical derivation of surprisal without importing Shannon entropy. Closed by replacing D7 with D7' (conditional Kolmogorov complexity K(B\|A)). BA-G1 demoted from Bridge Axiom to Compatibility Remark R-BA. The single remaining external dependency is I-KC (Kolmogorov framework from ZP-C), which is an import, not a bridge axiom. |
| **OQ-G2** | *Closed — ZP-H T-H1* | Left adjoint verification for instantiation functors. Resolved in ZP-H v1.0 by direct universal property verification for each functor. |
| **OQ-G3** | *Closed — ZP-H C-H1 through C-H4* | Explicit construction of four instantiation functors. Resolved in ZP-H v1.0. |
| **OQ-G4** | *Closed — ZP-H T-H2* | Reconciliation of categorical and ZP-C singularity characterizations. Resolved in ZP-H v1.0. Undefined domain (ZP-G) and infinite accumulation (ZP-C) shown to be the same obstruction under $F_C$. |
| **I-KC** | *Import — named dependency* | Conditional Kolmogorov complexity K(x\|y) and the coding theorem, imported from ZP-C D1 and standard algorithmic information theory. This is an import, not a bridge axiom: K(x\|y) is a fully defined mathematical object. ZP-G is no longer purely categorical; this dependency is explicitly stated. |
| **AX-G1** | *Axiom — intentional* | Asymmetry: initial object 0, no terminal object. Foundational structural commitment. Not a gap. |
| **AX-G2** | *Axiom — intentional* | Source asymmetry: $\hom(X, 0) = \varnothing$ for $X \not\cong 0$. Foundational irreversibility commitment. Not a gap. |
| **R-BA** | *Remark — BA-G1 demoted* | Leinster Shannon entropy characterization is now a compatibility remark, not a bridge axiom premise. Derivable from D7' and I-KC via the coding theorem. Not a gap. |

## X. Validation Status

| Component | Status / Notes |
|---|---|
| **D1: Category** | Valid — Definition. Standard; foundational. Unchanged. |
| **D2: Uniqueness notation** | Valid — Definition. Unchanged. |
| **D3: Initial object** | Valid — Definition. Load-bearing for T2, T7. Unchanged. |
| **D4: Terminal object** | Valid — Definition. Defined for exclusion. AX-G1 prohibits it. Unchanged. |
| **D5: Morphism chain** | Valid — Definition. Native to C. Unchanged. |
| **D6: Functor** | Valid — Definition. Standard. Unchanged. |
| **D7': Native categorical surprisal** | Valid — Definition [new in v1.1]. $K(x_B \| x_A)$ via I-KC. Replaces D7. Well-defined up to additive constant c. Structurally invariant. |
| **I-KC: Kolmogorov import** | Import — named [new in v1.1]. $K(x \| y)$ and coding theorem from ZP-C. Not a bridge axiom. Introduces explicit ZP-C dependency into ZP-G. |
| **AX-G1: Asymmetry Axiom** | Axiom — intentional. Unchanged. |
| **AX-G2: Source Asymmetry** | Axiom — intentional. Unchanged. |
| **R-BA: BA-G1 compatibility remark** | Remark — [BA-G1 demoted from Bridge Axiom in v1.0]. Shannon entropy functor compatible with D7' up to O(c) by coding theorem. No longer a premise of any theorem. |
| **T1: Uniqueness of initial object** | Valid — Derived. Unchanged. ✓ |
| **T2: Universal constituent** | Valid — Derived. Unchanged. ✓ |
| **T3: Unreachability of 0** | Valid — Derived. Unchanged. ✓ |
| **R1: Structural inversion** | Valid — Remark. Unchanged. |
| **T4: Chains are forward-only** | Valid — Derived. Unchanged. ✓ |
| **T5: Functors preserve initial objects** | Valid — Conditional on ZP-H T-H1 (closed). Unchanged. |
| **T6-a: Identity surprisal is zero** | Valid — Derived [new in v1.1]. $K(x_A \| x_A) = 0$ up to c. ✓ |
| **T6-b: Non-negative outward surprisal** | Valid — Derived [new in v1.1]. $K \geq 0$; strict for distinct objects. ✓ |
| **T6-c: Surprisal accumulates along chains** | Valid — Derived [new in v1.1]. Subadditivity of K. ✓ |
| **T6: Informational singularity** | Valid — Derived [rebuilt in v1.1]. Does not depend on BA-G1. Part II: undefined domain (AX-G2). Parts I, III: accumulation (T6-b, T6-c). ✓ |

| Component | Status / Notes |
|---|---|
| **T7: Categorical Zero Paradox** | Valid — Derived [Part IV strengthened in v1.1]. All six parts derived. BA-G1 not a dependency. ✓ |
| **OQ-G1: Native surprisal derivation** | Closed — D7', T6. No bridge axiom remains as a theorem premise. |

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