THE ZERO PARADOX

ZP-G: Category Theory

Version 1.3 | April 2026

Supersedes v1.2 | R2 added: categorical expression of ⊥ = {⊥} connecting note | All prior results unchanged

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This document is self-contained within category theory, with one explicit import from ZP-C: the conditional Kolmogorov complexity K(x|n) and the coding theorem connecting it to Shannon entropy. That import is named and labelled; it replaces the v1.0 Bridge Axiom BA-G1 (Leinster categorical entropy characterization) with a native derivation. All other content from ZP-A, ZP-B, ZP-D, and ZP-E remains excluded from this document. Cross-framework connections are deferred to ZP-H.

Honest labelling is the governing discipline. Every claim is marked as Axiom, Definition, Derived, Import, Design Commitment, or Remark. Nothing slides between categories.

Version 1.1 changes from v1.0: (1) D7 replaced by D7' (conditional Kolmogorov complexity K(B|A) as native categorical surprisal). (2) Import I-KC added: K(x|y) and the Kolmogorov coding theorem imported from ZP-C as a named dependency. (3) BA-G1 (Leinster Bridge Axiom) demoted to Remark R-BA: a compatibility result, no longer a theorem premise. (4) T6 rebuilt on D7'; proof is now self-contained within ZP-G plus the named ZP-C import. OQ-G1 closed. (5) All theorems from v1.0 that did not depend on BA-G1 (T1 through T5, T7) are unchanged in statement and proof.

Version 1.2 changes from v1.1: Theorem/Proposition/Lemma hierarchy applied throughout. T1 relabelled Proposition (subsidiary uniqueness result). T2, T3 relabelled Lemma (stepping-stone results for T6/T7). T4, T5 relabelled Proposition (derived but subsidiary). T6-a, T6-b relabelled Lemma (helpers for T6). T6-c relabelled Proposition (supports T6 but not the central claim). T6 and T7 remain Theorems — central claims of their sections. All statements and proofs are unchanged.

Version 1.3 changes from v1.2: Remark R2 added (Categorical Expression of Self-Containment). Connects initial object structure (T2 + AX-G2) to ZP-A CC-2 (⊥ = {⊥}). All prior results, axioms, and definitions unchanged.

Illustrated Companion: A paired ZP-G Illustrated Companion provides concrete examples and visual intuitions for the results here. Examples are kept separate from the formal layers to distinguish illustrative material from proofs.

Note on sequencing: The Zero Paradox framework labels its layers A through H, intentionally omitting F. ZP-G follows ZP-E directly; there is no missing document.

I. Categorical Primitives

Definitions D1 through D6 and results T1 through T5 are unchanged from v1.0. They are reproduced here for completeness and self-reference.

1.1 The Definition of a Category

Definition D1 — Category

Status: Definition — foundational

A category C consists of:

(i) Objects: A collection ob(C), written A, B, X, 0, ...

(ii) Morphisms: For each ordered pair (A, B), a collection hom(A, B) of morphisms, written f: A → B.

(iii) Composition: ∘: hom(B,C) × hom(A,B) → hom(A,C), written g ∘ f.

(iv) Identity: For each A, a morphism id_A: A → A.

Associativity: h ∘ (g ∘ f) = (h ∘ g) ∘ f.

Unit laws: id_B ∘ f = f = f ∘ id_A.

Definition D2 — Morphism Uniqueness Notation

Status: Definition

A morphism f: A → B is unique if for any g, h: A → B, g = h. Written: ∃! f: A → B.

1.2 Initial and Terminal Objects

Definition D3 — Initial Object

Status: Definition — load-bearing

An object 0 ∈ ob(C) is initial if for every X ∈ ob(C), there exists a unique morphism ι_X: 0 → X.

Proposition T1 — Uniqueness of the Initial Object

Status: Derived — standard category theory [unchanged from v1.0]

If 0 and 0' are both initial in C, ∃! isomorphism 0 □ 0'. The initial object is unique up to unique isomorphism.

Proof: ∃! f: 0 → 0' and ∃! g: 0' → 0. Initiality forces g ∘ f = id_0 and f ∘ g = $id_{0'}$. Therefore f is a unique isomorphism. ✓

Definition D4 — Terminal Object

Status: Definition — defined for exclusion

An object 1 ∈ ob(C) is terminal if for every X ∈ ob(C), ∃! morphism τ_X: X → 1. The Zero Paradox framework is not a zero object category (0 □ 1).

II. The Foundational Axioms of ZP-G

Axiom AX-G1 — Asymmetry Axiom

Status: Axiom — foundational structural commitment [unchanged from v1.0]

The category C possesses an initial object 0 and no terminal object.

Formally: ∃ 0 ∈ ob(C) satisfying D3. ¬∃ 1 ∈ ob(C) satisfying D4.

Correspondence: In ZP-A: join-semilattice without ⊤ and without ∧. In ZP-B: Q_2 has no element to which all paths converge. The present axiom is the categorical generalization.

Axiom AX-G2 — Source Asymmetry

Status: Axiom — foundational [unchanged from v1.0]

For any non-initial object X ≠ 0: hom(X, 0) = ∅.

Motivation: The categorical expression of irreversibility. Morphisms ι_X: 0 → X exist for all X. Their reversal does not exist. AX-G2 is consistent with AX-G1 but not derivable from it.

III. Universal Constituent and Unreachability

Lemma T2 — Universal Constituent

Status: Derived — from D3 and AX-G1 [unchanged from v1.0]

For every X ∈ ob(C), ∃! ι_X: 0 → X. The initial object 0 is the universal categorical source.

Proof: Immediate from D3 and AX-G1. ✓

Lemma T3 — Unreachability of 0

Status: Derived — from AX-G2 [unchanged from v1.0]

For any X ≠ 0: hom(X, 0) = ∅. The initial object 0 is unreachable from any non-initial object.

Proof: Direct from AX-G2. ✓

Remark R1 — Structural Inversion — The Categorical Zero Paradox

Status: Remark [unchanged from v1.0]

T2 and T3 together constitute the categorical Zero Paradox. 0 reaches every object (T2); no non-initial object reaches 0 (T3). This is not a logical contradiction. It is a structural inversion: the unique universal source is the unique object with no incoming non-trivial morphisms.

Remark R2 — Categorical Expression of Self-Containment

Status: Remark — connecting note to ZP-A CC-2 [new in v1.3]

Remark R2 — Categorical Expression of Self-Containment

R1 frames the structural inversion of 0. This remark connects that structure to ZP-A CC-2: $\bot = \{\bot\}$. The null state is its own extension — a Quine atom. A self-containing object has no external interpreter by structure; it IS its own interpretation.

In categorical terms, this corresponds to two conditions together: (1) AX-G2: $\hom(X, 0) = \varnothing$ for all $X \neq 0$ — no morphism can reach inside 0 from outside; and (2) T2: $\exists!\ \iota_X: 0 \to X$ for all X — 0 is structurally present in every object. Together these are the categorical image of undifferentiated self-containment: unreachable from without, yet constitutive of everything.

0 "points in all directions" (T2) because it is the undifferentiated ground from which all differentiation proceeds — not because it selects a direction. The uniqueness of each ι_X is not a choice among alternatives; it is the absence of internal structure that would allow differentiation among morphisms.

This remark bridges ZP-G to ZP-A CC-2. The formal correspondence between the categorical initial object structure and the set-theoretic Quine atom $\bot = \{\bot\}$ is made explicit in ZP-H.

IV. Monotone Structure and the Additive Ontology

Definition D5 — Morphism Chain

Status: Definition [unchanged from v1.0]

A morphism chain of length n from 0 is a sequence:

$0 = X_0 \to X_1 \to \ldots \to X_n$

where each $X_k \to X_{k+1}$ is a morphism in C.

Proposition T4 — Chains are Forward-Only

Status: Derived — from AX-G2 [unchanged from v1.0]

No morphism chain from 0 can return to 0 through non-initial objects.

Proof: A return morphism $X_n \to 0$ for $X_n \neq 0$ would contradict AX-G2. ✓

Definition D6 — Functor

Status: Definition — standard category theory [unchanged from v1.0]

A functor $F: C \to D$ consists of an object map $F: ob(C) \to ob(D)$ and a morphism map $F: \hom_C(A,B) \to \hom_D(F(A), F(B))$, preserving composition and identity.

Proposition T5 — Functors Preserve Initial Objects

Status: Derived — [OQ-G2 closed in ZP-H T-H1; unchanged from v1.0]

Proposition T5 — Functors Preserve Initial Objects

For each instantiation functor $F \in \{F_A, F_B, F_C, F_D\}$, F(0) is an initial object in the codomain. Verified by direct universal property check in ZP-H T-H1. ✓

V. The Kolmogorov Import from ZP-C

This section contains the single import from outside category theory that ZP-G v1.1 requires. It is named, scoped, and labelled explicitly. It replaces BA-G1 (the Leinster Bridge Axiom from v1.0) as the informational foundation of D7'.

Import I-KC — Conditional Kolmogorov Complexity from ZP-C

Status: Import — from ZP-C D1 and standard algorithmic information theory

What is imported: The conditional Kolmogorov complexity K(x|y), defined as the length of the shortest program p such that U(p, y) = x, where U is a fixed universal Turing machine:

$K(x|y) = \min \{ |p| : U(p, y) = x \}$

Key property — The Coding Theorem: For any computable probability measure P, Kolmogorov complexity and Shannon entropy are related up to an additive constant c (depending only on U, not on x or y):

$K(x|y) \approx -\log_2 P(x|y) + O(c)$

The coding theorem is a standard result of algorithmic information theory (Li and Vitanyi, An Introduction to Kolmogorov Complexity and Its Applications). It is not derived within ZP-G. It is imported as a named result.

Scope of import: I-KC imports K(x|y) and the coding theorem only. No other content from ZP-C is imported into ZP-G. The Kolmogorov complexity machinery is already present in ZP-C D1 (incompressibility threshold P_0), so I-KC introduces no new external dependency into the overall Zero Paradox framework — it only introduces a dependency within ZP-G specifically.

Computability note: K(x|y) is not computable in general (it is approximable from above by standard results). This is not a defect for the present purposes: the framework requires that K(x|y) be well-defined, not that it be computable. The ontological claims of ZP-G do not depend on computability.

Status: This is an import, not a bridge axiom. A bridge axiom is a claim that cannot be derived from either side and must be assumed. K(x|y) is a defined mathematical object with a complete internal theory. I-KC is a decision to use that object within ZP-G, not an assumption about it.

VI. Categorical Information Theory [Rebuilt in v1.1]

6.1 Native Categorical Surprisal — Definition D7'

Version 1.0 defined categorical surprisal via the Shannon entropy functor H imported through BA-G1. Version 1.1 replaces this with conditional Kolmogorov complexity, imported via I-KC. The definition is native to the morphism structure of C.

Definition D7' — Native Categorical Surprisal
Status: Definition — from D5, I-KC [replaces D7 from v1.0]
Let f: A → B be a morphism in C. Represent A and B as binary strings x_A and x_B via any injective encoding consistent with the morphism structure of C. The categorical surprisal of f is:
$I(f) = K(x_B \mid x_A)$
the conditional Kolmogorov complexity of the target given the source.
Interpretation: I(f) measures the minimum description length of B given knowledge of A. It is the irreducible informational content added by the transition f: A → B, independent of any probability distribution.
Well-definedness: $K(x_B\|x_A)$ depends on the encoding of objects as strings. Different encodings yield values differing by at most an additive constant c (the coding theorem constant of I-KC). All structural claims below are invariant under this additive constant: they concern whether I(f) is finite, zero, or undefined — not its precise numerical value.
Relationship to v1.0 D7: By the coding theorem (I-KC), $K(x_B\|x_A) \approx -\log_2 P(x_B\|x_A) + O(c)$ for any computable measure P. Therefore D7' and D7 are equivalent up to O(c). The choice of D7' over D7 is not a change in what is being measured; it is a change in how the measure is defined — natively versus via import.

6.2 Properties of the Native Surprisal

Lemma T6-a — Surprisal of the Identity Morphism is Zero
Status: Derived — from D7' and I-KC
Claim: $I(id_A) = K(x_A\|x_A) = 0$ up to the additive constant c.
Proof: The shortest program producing x_A given x_A is the empty program (output the input). Therefore $K(x_A\|x_A) = 0$ up to c. The identity morphism adds no informational content. ✓

Lemma T6-b — Surprisal is Non-Negative for Forward Morphisms
Status: Derived — from D5, D7', AX-G2
Claim: For any morphism f: A → B in a forward morphism chain from 0, I(f) ≥ 0 up to c, with strict inequality when A ≠ B.
Proof: $K(x_B\|x_A) \geq 0$ by definition (program length is non-negative). Strict inequality holds when x_B cannot be computed from x_A by the empty program — i.e., when A and B are distinct objects encoding distinct states. In a forward morphism chain (D5), each step adds content by the additive ontology (AX-G2). ✓

Proposition T6-c — Surprisal Accumulates Along Chains

Status: Derived — from D5, T6-b, subadditivity of K

Claim: For a morphism chain $0 = X_0 \to X_1 \to \ldots \to X_n$, the total surprisal $\sum I(X_k \to X_{k+1}) \geq 0$, with monotone accumulation as n increases.

Proof: By subadditivity of Kolmogorov complexity: $K(x_n|x_0) \leq \sum K(x_{k+1}|x_k) + O(n \cdot c)$. Each term is ≥ 0 by T6-b. Adding distinct objects strictly increases the total. ✓

VII. The Informational Singularity of 0 [Rebuilt in v1.1]

This is the central theorem of the information-theoretic section. In v1.0, it depended on BA-G1. In v1.1, it is proved from D7' and AX-G2 alone, with I-KC as the only external dependency.

Theorem T6 — Informational Singularity of 0

Status: Derived — from AX-G2, D7', I-KC [rebuilt from v1.0, OQ-G1 closed]

Setup: Let 0 be the initial object of C (AX-G1). Let $I(f) = K(x_B|x_A)$ be the categorical surprisal (D7'). Let I-KC provide the Kolmogorov framework.

Part I — Outward surprisal accumulates (from T6-b, T6-c): For any morphism chain $0 = X_0 \to \ldots \to X_n$, $\sum I(X_k \to X_{k+1}) \geq 0$, with strict accumulation as n increases. ✓

Part II — Inward surprisal is undefined (from AX-G2): For any X ≠ 0, hom(X, 0) = ∅ (AX-G2). Therefore D7' cannot be applied to any morphism f: X → 0 from outside 0 — no such morphism exists. I(X → 0) is undefined not because K diverges to infinity, but because there is no morphism to apply D7' to. The undefined-domain condition is strictly stronger than divergence. ✓

Part III — The singularity: 0 is the unique object in C for which outward surprisal is defined and accumulates (Part I) while inward surprisal is undefined by absence of morphisms (Part II). This is the informational singularity: the initial object is informationally accessible in the outward direction and categorically inaccessible in the inward direction. The singularity is structural, not numerical. It does not require K to diverge — it requires only AX-G2 and D7'. ✓

Comparison with ZP-C (to be reconciled in ZP-H T-H2): ZP-C establishes that the discrete surprisal DF diverges (numerically, to ∞) along infinite sequences approaching 0 in Q_2. T6 Part II establishes that I(X → 0) is undefined (domain-absent) for any X ≠ 0 in C. These are compatible: undefined is stronger than infinite. ZP-H T-H2 proves they describe the same obstruction under the functor F_C.

Status: DERIVED. Depends on AX-G1, AX-G2, D3, D5, D7', I-KC, T6-a, T6-b, T6-c. OQ-G1 is closed. BA-G1 is no longer a premise of T6. ✓

7.1 Compatibility with Shannon Entropy — BA-G1 Demoted to Remark

Remark R-BA — Compatibility of D7' with the Shannon Entropy Functor

Status: Remark — BA-G1 demoted from Bridge Axiom [v1.0] to Compatibility Remark [v1.1]

Version 1.0 introduced BA-G1 as a bridge axiom: it imported Leinster's categorical characterization of Shannon entropy (naturality, maximality, chain rule) to define the surprisal functor. BA-G1 was the only bridge axiom in ZP-G v1.0 and was the source of OQ-G1.

In v1.1, BA-G1 is no longer a premise of any theorem. It is retained here as a compatibility remark: the coding theorem (I-KC) guarantees that D7' and the Shannon functor of BA-G1 are equivalent up to an additive constant c. Specifically:

$K(x_B|x_A) \approx H(F(B)) - H(F(A)) + O(c)$

for any computable probability measure P consistent with the morphism structure of C. This means all quantitative results that v1.0 derived from BA-G1 remain valid under D7' — they differ only by the additive constant c, which does not affect any structural (finite/zero/undefined) claim.

BA-G1 is not false. It is not retired. It is now a derived compatibility result rather than an assumed premise. Any reader who finds the Shannon characterization more intuitive than Kolmogorov complexity may use BA-G1 as an equivalent formulation, knowing that I-KC and the coding theorem connect them.

VIII. The Categorical Zero Paradox — Formal Statement

Theorem T7 is the closing theorem of ZP-G. Its statement is unchanged from v1.0. Its proof is strengthened: Part IV (informational singularity) now rests on T6 as rebuilt in v1.1, which does not depend on BA-G1.

Theorem T7 — The Categorical Zero Paradox

Status: Derived — Closing Theorem [Part IV strengthened in v1.1]

Setup: Let C satisfy AX-G1 and AX-G2. Let I be the categorical surprisal from D7'. Let I-KC provide the Kolmogorov framework.

Part I — Universal Constituent (T2): $\forall X \in ob(C), \exists! \iota_X: 0 \to X$. The initial object 0 is the universal categorical source.

Part II — Unreachability (T3): $\forall X \neq 0$, $hom(X, 0) = \emptyset$. No non-initial object reaches 0.

Part III — Forward Irreversibility (T4): No morphism chain from 0 can return to 0 through non-initial objects.

Part IV — Informational Singularity (T6, rebuilt): $I(X \to 0)$ is undefined for all $X \neq 0$ (no such morphism exists, AX-G2). Outward surprisal from 0 accumulates along any morphism chain (T6-b, T6-c). 0 is an informational singularity: undefined inward, accumulating outward. This part no longer depends on BA-G1.

Theorem T7 — The Categorical Zero Paradox
Part V — The Structural Inversion: Parts I and II together constitute the paradox. 0 is the unique universal source of all objects, and simultaneously the unique object unreachable from outside. The foundation is the one object the morphism machinery cannot return to.
Part VI — Resolution: The paradox is not a logical contradiction. It is a structural inversion. The correct tools for characterizing 0 are the universal property (D3) and D7' applied to outward morphisms from 0. Under these tools, 0 is fully characterized. The paradox is the precise boundary between what can reach 0 and what cannot.
Status: DERIVED — Closing Theorem. Depends on D3, D5, D7', AX-G1, AX-G2, I-KC, T2, T3, T4, T6. BA-G1 is not a dependency. ✓

IX. Open Items Register for ZP-G v1.2

Item	Status	Description
OQ-G1	Closed — D7', T6	Native categorical derivation of surprisal without importing Shannon entropy. Closed by replacing D7 with D7' (conditional Kolmogorov complexity K(B|A)). BA-G1 demoted from Bridge Axiom to Compatibility Remark R-BA. The single remaining external dependency is I-KC (Kolmogorov framework from ZP-C), which is an import, not a bridge axiom.
OQ-G2	Closed — ZP-H T-H1	Left adjoint verification for instantiation functors. Resolved in ZP-H v1.0 by direct universal property verification for each functor.
OQ-G3	Closed — ZP-H C-H1 through C-H4	Explicit construction of four instantiation functors. Resolved in ZP-H v1.0.
OQ-G4	Closed — ZP-H T-H2	Reconciliation of categorical and ZP-C singularity characterizations. Resolved in ZP-H v1.0. Undefined domain (ZP-G) and infinite accumulation (ZP-C) shown to be the same obstruction under the functor F_C.
I-KC	Import — named dependency	Conditional Kolmogorov complexity K(x|y) and the coding theorem, imported from ZP-C D1 and standard algorithmic information theory. This is an import, not a bridge axiom: K(x|y) is a fully defined mathematical object. ZP-G is no longer purely categorical; this dependency is explicitly stated.
AX-G1	Axiom — intentional	Asymmetry: initial object 0, no terminal object. Foundational structural commitment. Not a gap.
AX-G2	Axiom — intentional	Source asymmetry: hom(X, 0) = ∅ for X ≠ 0. Foundational irreversibility commitment. Not a gap.

Item	Status	Description
R-BA	Remark — BA-G1 demoted	Leinster Shannon entropy characterization is now a compatibility remark, not a bridge axiom premise. Derivable from D7' and I-KC via the coding theorem. Not a gap.

X. Validation Status

Component	Status / Notes
D1: Category	Valid — Definition. Standard; foundational. Unchanged.
D2: Uniqueness notation	Valid — Definition. Unchanged.
D3: Initial object	Valid — Definition. Load-bearing for T2, T7. Unchanged.
D4: Terminal object	Valid — Definition. Defined for exclusion. AX-G1 prohibits it. Unchanged.
D5: Morphism chain	Valid — Definition. Native to C. Unchanged.
D6: Functor	Valid — Definition. Standard. Unchanged.
D7': Native categorical surprisal	Valid — Definition [new in v1.1]. $K(x_B\|x_A)$ via I-KC. Replaces D7. Well-defined up to additive constant c. Structurally invariant.
I-KC: Kolmogorov import	Import — named [new in v1.1]. $K(x\|y)$ and coding theorem from ZP-C. Not a bridge axiom. Introduces explicit ZP-C dependency into ZP-G.
AX-G1: Asymmetry Axiom	Axiom — intentional. Unchanged.
AX-G2: Source Asymmetry	Axiom — intentional. Unchanged.
R-BA: BA-G1 compatibility remark	Remark — [BA-G1 demoted from Bridge Axiom in v1.0]. Shannon entropy functor compatible with D7' up to O(c) by coding theorem. No longer a premise of any theorem.
Proposition T1: Uniqueness of initial object	Valid — Derived. Relabelled Proposition in v1.2 (subsidiary uniqueness result). Unchanged. ✓
Lemma T2: Universal constituent	Valid — Derived. Relabelled Lemma in v1.2 (stepping-stone result). Unchanged. ✓
Lemma T3: Unreachability of 0	Valid — Derived. Relabelled Lemma in v1.2 (stepping-stone result). Unchanged. ✓
R1: Structural inversion	Valid — Remark. Unchanged.
R2: Categorical expression of self-containment	Valid — Remark [new in v1.3]. Connects T2 + AX-G2 to ZP-A CC-2 ($\perp = \{\perp\}$). No new derivation; explanatory bridge note.

Component	Status / Notes
Proposition T4: Chains are forward-only	Valid — Derived. Relabelled Proposition in v1.2. Unchanged. ✓
Proposition T5: Functors preserve initial objects	Valid — Conditional on ZP-H T-H1 (closed). Relabelled Proposition in v1.2. Unchanged. ✓
Lemma T6-a: Identity surprisal is zero	Valid — Derived [new in v1.1]. Relabelled Lemma in v1.2. $K(x_A\|x_A) = 0$ up to c. ✓
Lemma T6-b: Non-negative outward surprisal	Valid — Derived [new in v1.1]. Relabelled Lemma in v1.2. $K \geq 0$; strict for distinct objects. ✓
Proposition T6-c: Surprisal accumulates along chains	Valid — Derived [new in v1.1]. Relabelled Proposition in v1.2. Subadditivity of K. ✓
Theorem T6: Informational singularity	Valid — Derived [rebuilt in v1.1]. Does not depend on BA-G1. Part II: undefined domain (AX-G2). Parts I, III: accumulation (T6-b, T6-c). ✓
Theorem T7: Categorical Zero Paradox	Valid — Derived [Part IV strengthened in v1.1]. All six parts derived. BA-G1 not a dependency. ✓
OQ-G1: Native surprisal derivation	Closed — D7', T6. No bridge axiom remains as a theorem premise.